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WHAT IS A QUANTUM FIELD THEORY?

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Introduction. Quantum field theory began in 1927 with a paper by Dirac [**1**] in which he sought a framework that would unite the classical electromagnetism of Maxwell with quantum mechanics. Ever since then it has been under continuous scrutiny by physicists, which illustrates the fact that in some ways, physics is more highly focused than mathematics. A mathematician is not forced to work on the Riemann hypothesis but ambitious physicists must work in the areas which have experimental durability. In view of all this attention from a different culture, it is not surprising that conventions and modes of reasoning in quantum field theory became difficult for mathematicians to penetrate.

In this article I shall survey a small part of the mathematical work accomplished in quantum field theory. My intent is to convince you that quantum field theories have, in the end, turned out to be very natural generalizations of Brownian motion and other "diffusions" which already occupy a central position in analysis and probability.

I will begin with one-dimensional quantum mechanics and show how there is a diffusion associated with it. In quantum mechanics the object is to determine the *wave function*

$$\Psi \equiv \Psi(t, Y).$$

For t fixed, Ψ belongs to $L^2(\mathbf{R})$ and is thought of as a time (t) dependent vector in that Hilbert space. Usually, one is given Ψ at some initial time, $t = 0$, and the problem is to determine Ψ at a different time by solving the *Schrödinger equation*

$$\left(\frac{1}{i}\frac{\partial}{\partial t} - H\right)\Psi = 0.$$

H is a partial differential operator called the *Hamiltonian*. It has the form

$$H = -\partial^2/\partial Y^2 + V$$

where V denotes multiplication by a function $V(Y)$. Provided some regularity conditions are imposed on V it is possible to solve this equation to obtain a trajectory in $L^2(\mathbf{R})$, $t \to \Psi(t)$, with value at $t = 0$ specified. By considering this process as a map

$$\Psi(t = 0) \overset{U_t}{\to} \Psi(t),$$

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