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THE DEVELOPMENT OF SQUARE FUNCTIONS IN THE WORK OF A. ZYGMUND

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I've decided to write this essay about "square functions" for two reasons. First, their development has been so intertwined with the scientific work of A. Zygmund that it seems highly appropriate to do so now on the occasion of his 80th birthday. Also these functions are of fundamental importance in analysis, standing as they do at the crossing of three important roads many of us have travelled by: complex function theory, the Fourier transform (or orthogonality in its various guises), and real-variable methods. In fact, the more recent applications of these ideas, described at the end of this essay, can be seen as confirmation of the significance Zygmund always attached to square functions.

This is going to be a partly historical survey, and so I hope you will allow me to take the usual liberties associated with this kind of enterprise: I will break up the exposition into certain "historical periods", five to be precise; and by doing this I will be able to suggest my own views as to what might have been the key influences and ideas that brought about these developments.

One word of explanation about "square functions" is called for. A deep concept in mathematics is usually not an idea in its pure form, but rather takes various shapes depending on the uses it is put to. The same is true of square functions. These appear in a variety of forms, and while in spirit they are all the same, in actual practice they can be quite different. Thus the metamorphosis of square functions is all important.

First period (1922–1926): The primordial square functions. It appears that square functions arose first in an explicit form in a beautiful theorem of Kaczmarz and Zygmund dealing with the almost everywhere summability of orthogonal expansions. The theorem was proved in 1926 as the culmination of several papers each had written at about that time. The theorem itself was an outgrowth of what certainly was one of the main preoccupations of analysts at that time, namely the question of convergence of Fourier series. The problem was the following. Suppose $f = f(\theta)$ is a continuous function on the circle, $0 \leqslant \theta \leqslant 2\pi$, or more generally assume that f is in $L^2(0, 2\pi)$ or even that f is merely integrable; then does its Fourier series

$$(1) \qquad \sum a_n e^{in\theta}, \quad \text{with } a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta}\, d\theta,$$

converge almost everywhere?

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