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A COMPUTER-ASSISTED PROOF OF THE FEIGENBAUM CONJECTURES

BY OSCAR E. LANFORD III[1]

1. Introduction. Let $\mathcal{M}$ denote the space of continuously differentiable even mappings ψ of the interval $[-1, 1]$ into itself such that

M1. $\psi(0) = 1$,

M2. $x\psi'(x) < 0$ for $x \neq 0$.

M2 says that ψ is strictly increasing on $[-1, 0)$ and strictly decreasing on $(0, 1]$, so $\mathcal{M}$ is a space of mappings which are unimodal in a strict sense.

Condition M1 says that the unique critical point 0 is mapped to 1. We want to consider ψ's which map 1 slightly – but not too far – to the left of 0. It may then be possible to find nonoverlapping intervals I_0 about 0 and I_1 near 1 which are exchanged by ψ. Technically, we proceed as follows: Write a for $-\psi(1) = -\psi^2(0)$ and b for $\psi(a)$; we suppress from the notation the dependence of a and b on ψ. Define $\mathcal{D}(\mathcal{T})$ to be the set of all ψ's in $\mathcal{M}$ such that:

D1. $a > 0$,

D2. $b > a$,

D3. $\psi(b) \leqslant a$.

The two intervals $I_0 = [-a, a]$ and $I_1 = [b, 1]$ are then nonoverlapping and ψ maps I_0 into I_1 and vice versa. If $\psi \in \mathcal{D}(\mathcal{T})$, then $\psi \circ \psi|_{I_0}$ has a single critical point, which is a minimum. By making the change of variables $x \longrightarrow -ax$, we replace I_0 by $[-1, 1]$ and the minimum by a maximum, i.e., if we define

$$\mathcal{T}\psi(x) = -\frac{1}{a}\,\psi \circ \psi(-ax) \quad \text{for } x \in [-1, 1]$$

then $\mathcal{T}\psi$ is again in $\mathcal{M}$. Thus, $\mathcal{T}$ defines a mapping of $\mathcal{D}(\mathcal{T})$ into $\mathcal{M}$. (In general, $\mathcal{T}\psi$ need not lie in $\mathcal{D}(\mathcal{T})$. If a is small, then $\mathcal{T}\psi(1)$ will be approximately 1 so $\mathcal{T}\psi$ will not satisfy D1. On the other hand, if $\psi(b)$ is near a, then $\mathcal{T}\psi(1)$ will be near -1 from which it follows that $\mathcal{T}\psi$ does not satisfy D2.)

M. Feigenbaum [6] has proposed an explanation for some universal features displayed by infinite sequences of period doubling bifurcations based on some conjectures about $\mathcal{T}$. We will not review has argument here; a version with due regard for mathematical technicalities may be found in Collet and Eckmann [3],

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