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THE CLOSURE OF THE SIMILARITY ORBIT OF A HILBERT SPACE OPERATOR

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1. Introduction. Let H be a complex separable infinite-dimensional Hilbert space and let $L(H)$ be the algebra of all (bounded linear) operators acting on H. The *similarity orbit* of $T \in L(H)$ is the subset

$$S(T) = \{WTW^{-1}: W \in L(H) \text{ is invertible}\}.$$

The purpose of this note is to announce the "almost complete" solution of the problem of characterizing the (norm) closure $S(T)^-$ of $S(T)$ in simple terms. Our results reduce the whole problem to the analysis of a very peculiar class of nilpotent operators and their compact perturbations. Complete results will appear elsewhere.

2. The main result. Assume that $A \in S(T)^-$, i.e., $\|A - W_nTW_n^{-1}\| \longrightarrow 0$ ($n \longrightarrow \infty$) for a suitable sequence $\{W_n\}_{n=1}^{\infty}$ of invertible operators. Since the spectrum of every operator in the sequence $\{W_nTW_n^{-1}\}_{n=1}^{\infty}$ coincides with the spectrum $\sigma(T)$ of T and, moreover, every single piece of $\sigma(W_nTW_n^{-1})$ (essential spectrum, left or right essential spectrum, normal eigenvalues, etc.) coincides with the corresponding piece of $\sigma(T)$, it is not difficult to see, by using the upper semicontinuity of separate parts of the spectrum (see, e.g., [5, Theorem 3.16]) that A necessarily satisfies

(0) $\sigma(A) \supset \sigma(T)$ and each component of $\sigma(A)$ intersects $\sigma(T)$.

Furthermore, if f is an analytic function defined on a neighborhood of $\sigma(A)$ and we define $f(A)$ via Riesz-Dunford functional calculus, then it is easily seen that $\|f(A) - f(W_nTW_n^{-1})\| \longrightarrow 0$ ($n \longrightarrow \infty$). If σ is a clopen subset of $\sigma(A)$, $f(\lambda) \equiv 1$ on a neighborhood of σ and $f(\lambda) \equiv 0$ on a neighborhood of $\sigma(A)\backslash\sigma$, then $P(\sigma; A) = f(A)$ is the Riesz' idempotent corresponding to σ [6, Chapter XIV]. Recall that $\lambda \in \sigma(A)$ is a *normal eigenvalue* if λ is an isolated point of $\sigma(A)$ and $\lambda \notin \sigma_e(A)$; equivalently: λ is an isolated point of $\sigma(A)$ and $P(\{\lambda\}; A)$ is a finite rank operator. (The set of all normal eigenvalues of A will be denoted by $\sigma_0(A)$.)

The continuity properties of the functional calculus imply that

(i) If $\lambda \in \sigma_0(A)$, then rank $P(\{\lambda\}; A) =$ rank $P(\{\lambda\}; T)$.

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