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FIXED POINT ALGEBRAS

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Although self-reference in arithmetic was used to impressive effect by Gödel in 1930 (published in 1931) when he noted the sentence asserting its own unprovability to be unprovable, and although this use immediately appealed to philosophers and philosophical logicians, it has largely been ignored by mathematical logicians. Indeed, it is only in the 1970s that arithmetic self-reference has begun to be systematically studied and applied. One aspect of this study is algebraic.

In simplest terms one can distinguish two types of self-reference—extensional and nonextensional. Extensional self-reference lends itself quite readily to algebraic description and modelling, with some types of extensional self-reference even being amenable to algebraic study. The purpose of the present paper is to expound upon this algebraic modelling, touching briefly on its successes and delineating roughly the limits to this success. The central notion of this exposition is that of a fixed point algebra. This notion is a new one—it is untested and, hence, of only provisional interest. But it does appear useful: It makes the present discussion cohere reasonably well; it provides a convenient framework in which to find and formulate questions about extensional self-reference; and it has allowed me to prove a theorem on the limitations of the use of finite algebras in studying such self-reference. Moreover, the algebra is fairly pleasant and provides a philosophically neutral framework in which to discuss arithmetic self-reference.

While the basic context from which the notion of a fixed point algebra arises is logical, the concept itself is algebraic and the following treatment is almost entirely algebraic. I have included logical material, including references to some rather arcane results of mathematical logic, in the discussion; but I have tried to keep this at a minimum. As a consequence, I declare most of the present paper accessible to the general mathematician who possesses only a small knowledge of boolean algebras. Only the logical asides (on background, motivation, and occasional applications) and the section on infinite fixed point algebras (for which I possess no nontrivial nonlogical examples) should be meaningless to the nonspecialist. [On the other hand, I must admit that, as I introduced fixed point algebras to serve as a vehicle for briefly surveying some of the algebraic aspects of self-reference in arithmetic, there isn't much left to the paper when the logic has been excised.]

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