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HYPERBOLIC GEOMETRY: THE FIRST 150 YEARS

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This will be a description of a few highlights in the early history of non-euclidean geometry, and a few miscellaneous recent developments. An Appendix describes some explicit formulas concerning volume in hyperbolic 3-space.

The mathematical literature on non-euclidean geometry begins in 1829 with publications by N. Lobachevsky in an obscure Russian journal. The infant subject grew very rapidly. Lobachevsky was a fanatically hard worker, who progressed quickly from student to professor to rector at his university of Kazan, on the Volga.

Already in 1829, Lobachevsky showed that there is a natural unit of distance in non-euclidean geometry, which can be characterized as follows. In the right triangle of Figure 1 with fixed edge a, as the opposite vertex A moves infinitely far away, the angle θ will increase to a limit θ_0 which is assumed to be strictly less than $\pi/2$. He showed that

$$a = -\log \tan(\theta_0/2)$$

if the unit of distance is suitably chosen. In particular,

$$a \approx (\pi/2) - \theta_0$$

if a is very small. (In the interpretation introduced by Beltrami forty years later, this unit of distance is chosen so that curvature $\equiv -1$.)

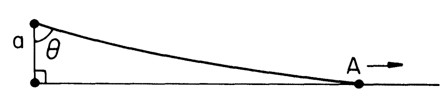

FIGURE 1. A right triangle in hyperbolic space

By early 1830, Lobachevsky was testing his "imaginary geometry" as a possible model for the real world. If the universe is non-euclidean in Lobachevsky's sense, then he showed that our solar system must be extremely small, in terms of this natural unit of distance. More precisely, taking the vertex A in Figure 1 to be the star Sirius and taking the edge a to be a suitably chosen radius of the Earth's orbit, he used the (unfortunately incorrect) estimate

$$\pi - 2\theta \cong 1.24 \text{ seconds of arc} \cong 6 \times 10^{-6} \text{ radians}$$

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