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POINCARÉ RECURRENCE AND NUMBER THEORY

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Introduction. Poincaré is largely responsible for the transformation of celestial mechanics from the study of individual solutions of differential equations to the global analysis of phase space. A system of differential equations such as those which embody the laws of Newtonian mechanics generates a one-parameter group of transformation of the manifold that represents the set of states of a dynamical system. The evolution of the dynamical system in time corresponds to a particular solution of the system of differential equations; it also corresponds to an orbit of the group of transformations acting on a single state. The efforts of the classical analysts in celestial mechanics had been directed to extracting by analytical means as much information as possible about the individual solutions to the system of differential equations. Poincaré's work gave impetus to a global approach which studies the totality of solutions and shifts attention to the transformation group of phase space.

Two of Poincaré's achievements which can be traced to this point of view are his theory of periodic solutions and his recurrence theorem. In the former of these, the topological nature of the phase space plays a key role; Poincaré showed how an essentially topological analysis of orbits, under certain conditions, can be used to establish the existence of a periodic solution curve. In his recurrence theorem, Poincaré demonstrated how measure-theoretic ideas, particularly the idea of a measure-preserving group of transformations, lead to the existence of numerous "approximately periodic"—or, "recurrent"—solution curves. The impact of these ideas is felt today in the establishment of two new disciplines: topological dynamics and ergodic theory. In topological dynamics one abstracts from the classical setup the topological space representing the totality of states of a dynamical system, together with the group of homeomorphisms corresponding to the evolution of the system from its position at time 0 to its position at time t. For ergodic theory, the phase space is replaced by an abstract measure space and the "dynamics" come from the action of a group of measure-preserving transformations of the measure space.

Initially these abstract settings for dynamical theory were contemplated in order to shed light on classical dynamics by focusing on those aspects of dynamical systems that were pertinent to the phenomena being studied. At the same time, however, the scope of dynamics was considerably broadened by the generality of the new theory, and the groundwork was laid for

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