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THE THEORY OF RECURSIVE FUNCTIONS, APPROACHING ITS CENTENNIAL[1]

(*Elementarrekursiontheorie vom höheren Standpunkte aus.*[2])

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ABSTRACT. An algorithm is a procedure, given by a finite set of instructions, to serve as follows in relation to a given infinite class of questions. (a) If we select any question from the class, the instructions will tell us how to perform a step. (b) After any step, if we do not receive the answer then, the instructions together with the existing situation will tell us what step to take next. (c) The instructions will enable us to recognize when a situation is reached in which the answer is before us, and to read it off then; and this will eventually happen if the question has an answer. In "steps" and "situations", what are we handling? Since there must be no ambiguity, surely some kind of regular complexes of occurrences of symbols from a given finite list. Such complexes can be coded by positive integers. Consider specifically an algorithm for computing a functional $\phi(\Theta; \mathfrak{A})$ where Θ are number-theoretic function variables and $\mathfrak{A}$ are natural number variables. A question is selected from the infinite class "What is the value of $\phi(\Theta; \mathfrak{A})$?" by specifying $(\Theta; \mathfrak{A})$. I gain some space for maneuver by using instead $(\Theta; \mathfrak{A}, 0)$. After any step, the situation will be represented by $(\Theta; \mathfrak{A}, b)$ where b is the code for the complex of symbols in it. By (c), there must be a functional $\chi(\Theta; \mathfrak{A}, b)$, for which we already know how to get the values, such that, in the situation represented by $(\Theta; \mathfrak{A}, b)$, $\chi(\Theta; \mathfrak{A}, b) = 0$ if the answer is not before us, and otherwise $\chi(\Theta; \mathfrak{A}, b) = \phi(\Theta; \mathfrak{A}, b) + 1$ where $\phi(\Theta; \mathfrak{A}, b)$ is the answer; and $\chi(\Theta; \mathfrak{A}, 0) = 0$. By (b) and (a), there must likewise be a functional $\rho(\Theta; \mathfrak{A}, b)$ such that in the situation represented by $(\Theta; \mathfrak{A}, b)$, if $\chi(\Theta; \mathfrak{A}, b) = 0$ then $(\Theta; \mathfrak{A}, \rho(\Theta; \mathfrak{A}, b))$ represents the situation after the next step; and $(\Theta; \mathfrak{A}, \rho(\Theta; \mathfrak{A}, 0))$ represents the situation after the first step. Now, putting $\phi(\Theta; \mathfrak{A}, b) \simeq \phi(\Theta; \mathfrak{A}, \rho(\Theta; \mathfrak{A}, b))$ if $\chi(\Theta; \mathfrak{A}, b) = 0$, $\simeq \chi(\Theta; \mathfrak{A}, b) \dot{-} 1$ if $\chi(\Theta; \mathfrak{A}, b) > 0$, we have a definition of $\phi(\Theta; \mathfrak{A}, b)$ of the form $\phi(\Theta; \mathfrak{A}, b) \simeq \psi(\lambda\mathfrak{A}b\phi(\Theta; \mathfrak{A}, b), \Theta; \mathfrak{A}, b)$ as in Kleene's first recursion theorem [**1952**, p. 348]; and $\phi(\Theta; \mathfrak{A}) \simeq \phi(\Theta; \mathfrak{A}, 0)$. Thence it is argued that the first recursion theorem, in a proper setting, enables all functionals $\phi(\Theta; \mathfrak{A})$ to be defined for which there are algorithms; and consequences are deduced therefrom.

The theory of recursive functions is nearly one hundred years old. For nearly the first fifty years it was the theory of what are now called "primitive

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[1] I prepared this paper for a general audience. Then I was amazed to find on the program twenty-three other papers (including four special sessions organized by Richard A. Shore) on recursion theory, which illustrates the health of the nonagenarian.

[2] Kleene is a North German form of Klein.