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THE MEANING OF MASLOV'S ASYMPTOTIC METHOD: THE NEED OF PLANCK'S CONSTANT IN MATHEMATICS

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ABSTRACT. H. Poincaré defined asymptotic expansions. Their use by the W. K. B. method introduced a new kind of solution of linear differential equations. Maslov showed their singularities to be merely apparent. The clarification of those results leads to the introduction of "Lagrangian functions", of their scalar product and of "Lagrangian operators", which constitutes a new structure: the "Lagrangian analysis". The last step of its definition requires the choice of a constant. That constant has to be Planck's constant, when the equation is the Schrödinger or the Dirac equation describing the hydrogen atom—the study of atoms with several electrons is very incomplete.

1. Henri Poincaré's main field, more precisely the one where the number of his publications is the highest, happens to be celestial mechanics. For instance, he tried to establish the convergence of the series by means of which the motion of the solar system is computed; it was a failure. He proved indeed the opposite: the divergence of those series, whose numerical values furnished the most impressive, precise and famous predictions in science during the last century! Henri Poincaré explained that paradox: those series give a very good approximation of the wanted result, provided only their first terms, namely, a reasonable number of them, are taken into account. Of course, demanding mathematicians to be reasonable is dubious but Henri Poincaré [4] made it clear by defining *the asymptotic expansion* $\sum_{n=0}^{\infty} a_n x^n$ of a function of x at the origin: it is a *formal series* such that for each natural number N there exists a positive number c_N such that

$$\left| f(x) - \sum_{n=0}^{N} a_n x^n \right| \leqslant c_N |x|^{N+1} \quad \text{for } x \text{ near } 0. \tag{1.1}$$

Thus an asymptotic expansion of f is a formal series able to give a very good approximation of $f(x)$, when x is small, but unable to supply the exact value of $f(x)$.

2. The W.K.B. method constructs *asymptotic solutions* of a linear differential equation

$$H\left(x, \frac{1}{\nu}\frac{\partial}{\partial x}\right)u(\nu, x) = 0 \qquad \left(x \in X = \mathbf{R}^l;\ \nu \in i\,[0, \infty[\right), \tag{2.1}$$

whose unknown is the function u and whose parameter ν *tends to* $i\infty$.

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