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# A UNIVERSAL DIFFERENTIAL EQUATION

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*Dedicated to the Memory of Walter Strodt*

THEOREM. *There exists a nontrivial fourth-order algebraic differential equation*

$$* \qquad P(y', y'', y''', y'''') = 0,$$

*where $P$ is a polynomial in four variables, with integer coefficients, such that for any continuous function $\varphi$ on $(-\infty, \infty)$ and for any positive continuous function $\epsilon(t)$ on $(-\infty, \infty)$, there exists a $C^\infty$ solution $y$ of $*$ such that*

$$|y(t) - \varphi(t)| < \epsilon(t) \quad \textit{for all } t \in (-\infty, \infty).$$

*One such specific equation (homogeneous of degree seven, with seven terms of weight* 14) *is*

$$3y'^4y''y''''^2 - 4y'^4y'''^2y'''' + 6y'^3y''^2y'''y'''' + 24y'^2y''^4y'''' - 12y'^3y''y'''^3 - 29y'^2y''^3y'''^2 + 12y''^7 = 0.$$

REMARK 1. From the proof, it will be clear that we can in addition ensure that $y(t_j) = \varphi(t_j)$ for any sequence $(t_j)$ of distinct real numbers such that $|t_j| \to \infty$ as $j \to \infty$.

REMARK 2. We may moreover make $y$ monotone if $\varphi$ is monotone.

REMARK 3. Without changing the equation $*$, if $\varphi$ and $\epsilon$ are only defined on an open interval $I$, then we can make $|y(t) - \varphi(t)| < \epsilon(t)$ for all $t \in I$, where $y$ is a $C^\infty$ solution of $*$ on $I$.

If we regard the uniform limits of solutions of $*$ as "weak solutions" (the way $y = |t|$ is a weak solution of $yy' - t = 0$ as the limit of $(t^2 + \epsilon^2)^{1/2}$ as $\epsilon \to 0$), then a corollary of our Theorem is that every continuous function $\varphi$ is a weak solution of $*$.

This Theorem may be regarded as an analogue, for analog computers, of the Universal Turing Machine (see [**R**, p. 23]), because of a theorem of Shannon (see [**S**, Theorem II]) that identifies the outputs of analog computers with the solutions of algebraic differential equations. A later paper of Pour-El requires some

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