BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 1, 1981

0002-9904/81/0000-0003/$01.75

BOOK REVIEWS

Hilbert's fourth problem, by A. V. Pogorelov, Wiley, New York; Holt, Rinehart and Winston, New York, 1979, vi + 97 pp., $16.00.

Hilbert's Problem 4 stripped of its comments[1] is this: Omit from his axioms for the foundations of geometry besides the parallel axioms all those which contain the concept of angle, and replace them by the triangle inequality, which follows from the congruence axiom for triangles (*CT*).

(1) Determine all geometries satisfying these conditions.

(2) Study the individual ones.

This is not quite Hilbert's formulation because with respect to angles he only omits the *CT*. The remaining angle axioms have no significant applications without *CT*. Pogorelov uses the preferable form given above.

Only two such geometries, besides the elementary ones, were known at the time, the Minkowskian satisfying the euclidean parallel axiom and Hilbert's geometry generalizing in a similar way the hyperbolic situation. It seems that Hilbert did not think of a mixed situation like a half plane.

Nor is it clear, whether he wanted to include nonsymmetric (n.s.) distances (a Minkowski metric may be symmetric or not). Hilbert was admittedly probing and considered, in fact, an analysis of the distance concept as one of the tasks connected with the problem. Since absence of symmetry violates one of the axioms Pogorelov does not admit n.s. distances.

That Hilbert was interested in them is evident from Hamel's thesis [2], which he directed immediately after his lecture. It dwells largely on n.s. distances. They were deemphasized in the later version [3] of [2], most probably because from the great variety of Desarguesian metrics (both symmetric and not) which Hamel exhibited, no appealing new one emerged. The situation changed in 1929, when Funk discovered a very interesting, always n.s. geometry which resembles euclidean geometry in some respects and hyperbolic in others. In addition it led to the proper definition of completeness: The balls $\{x|px \leqslant \rho\}$ are compact, but not necessarily the $\{x|xp \leqslant \rho\}$, where xy is the distance.

Hamel approached the problem through the Weierstrass Theory of the calculus of variations, which requires smoothness properties alien to the foundations of geometry, the framework visualized by Hilbert for the problem. This is less surprizing than it appears because Hilbert was very interested

[1]These are translated in *Hilbert's mathematical problems* **8** (1902) of this Bulletin. The translation is reprinted in [1], a report on the symposium on Hilbert's Problems held in 1974. Problem 4 appears there on pp. 131–141 with the title *Desarguesian spaces*. Hilbert's heading, slightly modernized, is: *The geometries in which the ordinary lines provide the shortest connections.* We refer the reader to the article in [1] for all facts concerning Problem 4 omitted here, because they are not connected to Pogorelov's book.