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AFFINE MANIFOLDS AND SOLVABLE GROUPS

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Let M be a *compact* affine manifold. Thus M has a distinguished atlas whose coordinate changes are locally in $\mathrm{Aff}(E)$, the group of affine automorphisms of Euclidean n-space E. Assume M is connected and without boundary.

The universal covering $\widetilde{M}$ of M has an affine immersion D: $\widetilde{M} \longrightarrow E$ which is unique up to composition with elements of $\mathrm{Aff}(E)$. Corresponding to D there is a homomorphism α: $\pi \longrightarrow \mathrm{Aff}(E)$, where π is the group of deck transformations of $\widetilde{M}$, such that D is equivariant for α. Set $\alpha(\pi) = \Gamma$. Let L: $\mathrm{Aff}(E) \longrightarrow GL(E)$ be the natural map.

THEOREM 1. *If Γ is nilpotent the following are equivalent*:

(a) *M is complete, i.e. D: $\widetilde{M} \longrightarrow E$ is bijective*;

(b) *D is surjective*;

(c) *no proper affine subspace of E is invariant under Γ*;

(d) *$L(\Gamma)$ is unipotent*;

(e) *M has parallel volume, i.e. $L(\Gamma) \subset SL(E)$*;

(f) *M is affinely isomorphic to $\Gamma\backslash G$ where G is a connected Lie group with a left-invariant affine structure and $\Gamma \subset G$ is a discrete subgroup*;

(g) *each de Rham cohomology class of M is represented by a differential form whose components in affine charts are polynomials.*

For abelian Γ the equivalence of (a), (d), and (e) is due to J. Smillie. We conjecture that (a), (b), (e), and (g) are equivalent even without nilpotence (if M is orientable). In general (a) $\Rightarrow$ (c) and (e) $\Rightarrow$ (c); but (c) $\not\Rightarrow$ (a) even for Γ solvable and M three-dimensional.

THEOREM 2. *The following are equivalent*:

(i) *M is finitely covered by a complete affine nilmanifold M_1 (i.e. conditions* (a) *through* (g) *of Theorem 1 hold for M_1)*;

(ii) *all eigenvalues of elements of $L(\Gamma)$ have norm* 1;

(iii) *M has a Riemannian metric whose coefficients in affine charts are polynomials.*

L. Auslander has conjectured that if M is complete then $\pi = \Gamma = \pi_1(M)$ is virtually solvable (i.e. contains a solvable subgroup of finite index); see [M] for discussion. This conjecture is true in dimension three (see [FG]).

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