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SUFFICIENCY OF McMULLEN'S CONDITIONS FOR f-VECTORS OF SIMPLICIAL POLYTOPES

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For convex d-polytope P let $f_i(P)$ equal the number of faces of P of dimension i, $0 \leqslant i \leqslant d-1$. $f(P) = (f_0(P), \ldots, f_{d-1}(P))$ is called the *f-vector of P*. An important combinatorial problem is the characterization of the class of all f-vectors of polytopes, and in particular of simplicial polytopes (i.e. those for which each facet is a simplex). McMullen in [5] conjectures a set of necessary and sufficient conditions for $(f_0, \ldots, f_{d-1})$ to be the f-vector of a simplicial d-polytope and proves this conjecture in the case of polytopes with few vertices. We sketch here a proof of the sufficiency[3] of these conditions, and derive in a related way a general solution to an upper bound problem posed by Klee.

The f-vectors of simplicial d-polytopes satisfy the *Dehn-Sommerville equations*

$$\sum_{i=j}^{d-1} (-1)^i \binom{i+1}{j+1} f_i(P) = (-1)^{d-1} f_j(P), \qquad -1 \leqslant j \leqslant d-1,$$

where we put $f_{-1}(P) = 1$. As in [6, p. 170], for d-vector $f = (f_0, \ldots, f_{d-1})$ and integer $e \geqslant d$ let

$$g_j^{(e)}(f) = h_{j+1}^{(e)}(f) = \sum_{i=-1}^{j} (-1)^{j-i} \binom{e-i-1}{e-j-1} f_i, \qquad -1 \leqslant j \leqslant e-1,$$

with the convention that $f_{-1} = 1$ and $f_i = 0$ for $i < -1$ or $i > d-1$. We note here that these relations are invertible, allowing us to express the f_i as nonnegative linear combinations of the $h_j^{(e)}(f)$. The Dehn-Sommerville equations for f are, for any $e \geqslant d$, equivalent to $g_i^{(e)}(f) = (-1)^{e-d} g_{e-i-2}^{(e)}(f)$, $-1 \leqslant i \leqslant [e/2] - 1$. Let h and i be positive integers. Then h can be written uniquely as

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[2]Supported, in addition, by an NSF Graduate Fellowship.

[3]ADDED IN PROOF. R. Stanley has proved necessity since this was written.