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ON NONLINEAR DESINGULARIZATION

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"Nonlinear desingularization" is an interesting, hitherto little studied phenomenon in nonlinear elliptic partial differential equations. By nonlinear desingularization we mean that a linear boundary value problem whose solution possesses one or more isolated singularities is a degenerate form of a family of nonlinear problems whose solutions are smooth, and moreover converge to the singular solution of the linear system upon degeneration. Such phenomena occur commonly in various branches of theoretical physics. For example, in Helmholtz's famous study of vortex motion of ideal fluids [1], a circular vortex filament is used to approximate a steady vortex ring of small cross section. In that case the Stokes stream function of the vortex filament is the Green's function for the axisymmetric Laplace operator, and hence has a singularity, while the Stokes stream function for the vortex ring of small cross-section satisfies a nonlinear elliptic partial differential equation, and is smooth [2]. Similar phenomena occur in the plasma physics of Tokomaks governed by the Lundquist equations, and in the onset of the "mixed state" [3] in type II superconductors associated with the Ginzberg-Landau equations. Moreover in [4], S. Adler has discussed how the nonlinear desingularization process leads to new classes of static Euclidean $SU(2)$ Yang-Mills fields (monopoles) with unit Pontrajagin index.

Mathematically speaking, nonlinear desingularization is a rather novel kind of bifurcation phenomenon involving parameter-dependent nonlinear problems. Indeed, we shall indicate in §3 below how our ideas on bifurcation, contained in [5], have an exact parallel in a typical situation of nonlinear desingularization.

1. The nonlinear problem and its linear degenerate form. Let Ω be a bounded domain in the plane $\mathbf{R}^2$, with boundary $\partial\Omega$, and let L be a formally self-adjoint, uniformly elliptic, second order operator, with smooth coefficients $a_{ij}(x)$ defined on Ω with the matrix $(a_{ij}(x))$ positive definite and

$$Lu = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_1}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right).$$

Then we consider the free boundary problem of finding a piecewise smooth

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