BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 5, September 1979

0002-9904/79/0000-0408/$01.75

BOOK REVIEWS

Symplectic groups, by O. T. O'Meara, Mathematical Surveys, no. 16, Amer. Math. Soc., Providence, R. I., 1978, xi + 122 pp., $22.80.

The isomorphism theory of the classical groups over fields was initiated fifty years ago when Schreier and van der Waerden [7] determined the automorphisms of the projective special linear groups.

In 1951 Dieudonné [3] published an extensive study of the automorphisms of the classical groups and some of their large subgroups over possibly noncommutative fields. His method is to consider the involutions in a classical group G over a sfield F and the manner in which an automorphism Λ of G transforms these involutions among themselves. Some of these involutions are then related, either directly or in combinations, to one-dimensional subspaces in the underlying vector space V on which the group G acts, that is, to points in the projective space $\mathbf{P}(V)$ of V. Thus the automorphism Λ is used to set up a bijective correspondence between the points in $\mathbf{P}(V)$ where, in the simplest case, the point associated with the involution σ corresponds to the point associated with $\Lambda\sigma$. Then, after establishing that this correspondence is a projectivity, the Fundamental Theorem of Projective Geometry can be used to show that the automorphism is of some standard type such as $\Lambda\colon \varphi \mapsto \chi(\varphi)g\varphi g^{-1}$ for $\varphi \in G$, where χ is a homomorphism from G into its center and g is a semilinear automorphism of V. Variations of this technique have been used by other authors but it is limited to those classical groups and subgroups with an abundant supply of involutions. The situation circa 1963 has been summarized by Dieudonné [4].

To handle groups with few involutions, O'Meara in 1967 introduced a new approach where the points of the projective space $\mathbf{P}(V)$ are now associated with a different type of object in the group G. For example, if G is linear or symplectic, a point can be associated with a nontrivial group of projective transvections in the following manner. A transvection is a linear transformation of V of the form $\tau = \tau(a, \rho)\colon x \mapsto x + \rho(x)a$ where $a \in V$ and $\rho \in \operatorname{Hom}(V, F)$ with $\rho(a) = 0$; when $\tau \neq 1$, the residual space $(\tau - 1)V$ is the line Fa in V. It is easier to work in the projective group PG and here the point Fa in $\mathbf{P}(V)$ is associated with the group of all projective transvections in PG with residual line Fa (plus the identity). The central problem now is to give a characterization of these groups of projective transvections and then deduce that their image under an automorphism Λ is of the same form, so that one can set up a projectivity in $\mathbf{P}(V)$ and determine the nature of Λ. In orthogonal groups there may be no transvections so these are replaced by plane rotations; one-dimensional subspaces of V are then obtained by intersecting planes. This new approach has been extremely successful, and in a sequence of papers over the last twelve years, O'Meara and his colleagues have developed an extensive theory of isomorphisms between subgroups of classical groups over fields and, more generally, studied arithmetic questions for congruence subgroups defined over integral domains. Naturally, there has