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TWO REDUCTIONS OF THE POINCARÉ CONJECTURE

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ABSTRACT. We study two reductions of the Poincaré conjecture. The first is group theoretic and is an improvement over Papakyriakopoulos' reduction [5]. The second reduces the conjecture to a special case of it.

We first examine Papakyriakopoulos' reduction and improve it. The method also gives a new proof of a crucial theorem in his reduction.

P. 1. CONJECTURE. Let G_p: $\{a_1, b_1, \ldots, a_p, b_p; \Pi_{i=1}^p [a_i, b_i], p > 1\}$ and let $Q_p = \{a_1, b_1, \ldots, a_p, b_p; \Pi_{i=1}^p [a_i, b_i], [a_1, b_1\tau]\}$, where $\tau \in [\Phi_p, \Phi_p]$, Φ_p being the free group generated by $a_1, b_1, \ldots, a_p, b_p$. Let T_p be an orientable surface of genus p and identity $\pi_1(T_p)$ with G_p. Then

(a) Q_p is torsion-free.

(b) The cover of T_p corresponding to the Kernel of the natural map φ_p: $G_p \longrightarrow Q_p$ is planar.

E. S. Rapaport proved [7] P.1.(a) and Papakyriakopoulos showed that P.1 implies the Poincaré conjecture [5]. He also considered the question ([5], [6]) whether P.1.(b) is group theoretic. Consider the following

P.2. CONJECTURE. The group Q_p defined above is a nontrivial free-product.

We will show

A. THEOREM. P.1 $\Rightarrow$ P.2 $\Rightarrow$ *Poincaré conjecture. Moreover,* P.1 *is group-theoretic.*

The crucial step in the reduction [5] of Poincaré conjecture to P.1 is a theorem which connects the problem of finding nontrivial simple loops in a certain normal subgroup with regular planar covers subordinate to it. This result was strengthened by Maskit in [4]; Lemmas 1 and 2 below imply his theorem. These lemmas connect the approach of Papakyriakopoulos with that of Stallings in [8].

Let

$$\{e\} \longrightarrow L \xrightarrow{i} G \xrightarrow{\varphi} H \longrightarrow \{e\}$$

be an exact sequence of groups, where G is the fundamental group of a closed

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