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STOCHASTIC INTEGRATORS

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ABSTRACT. The most general reasonable stochastic integrator is a semimartingale. For a large class of integrands the stochastic integral can be evaluated pathwise.

The notion of a semimartingale is a notion *ad hoc*. It is the result of an effort to generalize to the utmost the two known *techniques* of stochastic integration: If $Z = M + V$ is a decomposition of the semimartingale Z into a local martingale M and a process V of finite variation then one can define $\int_0^t X\,dZ$ as $\int_0^t X\,dM + \int_0^t X\,dV$. The second summand is an ordinary Stieltjes integral taken pathwise, while the first one is defined by Ito's technique.

The question arises whether this is the best one can do. More precisely, for which processes Z can one define $\int \cdot\, dZ$ in such a way that the integral has "reasonable" properties? Somewhat disappointingly, the theorem below states that every reasonable stochastic integrator is a semimartingale. This might come as a surprise in view of the very modest criterion of reasonableness adopted.

On the positive side, the proof of the theorem yields an equivalent definition of the integral which obviates the need to split the integrator Z as $Z = M + V$ and which lends itself to a *pathwise* computation of $\int X\,dZ$ for a large class of integrands X.

Stating our criterion of reasonableness requires some notation. Underlying everything is a complete probability space $(\Omega, \mathcal{G}, P)$ equipped with a filtration $\mathcal{F} = (\mathcal{F}_t\colon 0 \leqslant t < \infty)$ that has the usual properties [5], [7]. Let $\mathcal{T}$ denote the collection of all stopping times that take only finitely many values, each of them finite; let $\mathcal{A}$ denote the ring of subsets of the base space $B = \Omega \times [0, \infty)$ generated by the stochastic intervals $((S, T]]$, $S \leqslant T$ in $\mathcal{T}$; and denote by $\mathcal{R}$ the vector lattice of step functions over $\mathcal{A}$. Now if $Z\colon B \longrightarrow \mathbf{R}$ is any process then $dZ(((S, T]]) := Z_T - Z_S$, extended by linearity, defines a linear map

$$dZ\colon\ \mathcal{R} \longrightarrow L^0(\Omega, \mathcal{G}, P).$$

The following definition spells out our criterion of reasonableness.

DEFINITION. An adapted process Z is an L^p-integrator, $0 \leqslant p < \infty$, provided

(A_0) if A_n is a decreasing sequence in $\mathcal{A}$ with void intersection then

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