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MAXIMAL FUNCTIONS: A PROOF OF A CONJECTURE OF A. ZYGMUND

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In $\mathbf{R}^n$ let us consider the family B_n of parallelepipeds with sides parallel to the coordinate axes. We may ask for conditions upon the locally integrable function f in order that

$$[*] \qquad \lim_{\substack{x\in R\in B_n \\ \operatorname{diam}(R)\to 0}} \frac{1}{\mu\{R\}}\int_R f(y)\,d\mu(y) = f(x)$$

a.e. x., where μ = Lebesgue measure in $\mathbf{R}^n$.

In 1935 B. Jessen, J. Marcinkiewicz and A. Zygmund [1] showed that [*] holds so long as $f \in L(1 + (\log^+ L)^{n-1})(\mathbf{R}^n)$ locally. Furthermore this result is the best possible in the following sense: if $\psi(t)$ is an Orlicz's space defining function such that $\psi(t) = o(t(\log t)^{n-1})$, $t \to \infty$, then statement [*] is false for a typical L_ψ-function (typical in the sense of Baire's category). Of course the case $n = 1$ was known before as Lebesgue's Differentiation theorem.

The following natural problem was proposed by A. Zygmund: given a positive function Φ on $\mathbf{R}^2$, monotonic on each variable separately, let us consider the differentiation basis B_Φ in $\mathbf{R}^3$ defined by the two parameters family of parallelepipeds whose sides are parallel to the rectangular coordinate axis and whose dimensions are given by $s \times t \times \Phi(s, t)$, s, t positive real numbers. For which locally integrable functions f is statement [*] true with respect to the family B_Φ?

In general the differentiation properties of B_Φ must be, at least, not worse than B_3, the basis of all parallelepipeds in $\mathbf{R}^3$ whose sides have the direction of the coordinate axes, and, of course, not better than B_2. A. Zygmund conjectured after his 1935 paper that B_Φ behaves like B_2. This conjecture is now a theorem with applications to a.e. convergence of Poisson Kernels associated to certain symmetric spaces.

THEOREM. (a) *B_Φ differentiates integrals of functions which are locally in* $L(1 + \log^+ L)(\mathbf{R}^3)$, that is

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