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THE INTEGRABILITY PROBLEM FOR LIE EQUATIONS

BY HUBERT GOLDSCHMIDT[1]

The study of pseudogroups and geometric structures on manifolds associated to them was initiated by Sophus Lie and Élie Cartan. In comparing two such structures, Cartan was led to formulate the equivalence problem and solved it in the analytic case using the Cartan-Kähler theorem (see [2] and [3]). In the early 1950's, Spencer elaborated a program to study structures on manifolds, in particular complex structures, and their deformations in terms of partial differential equations. He set up the integrability problem for Lie pseudogroups and introduced partial differential equations which are equivalent to those solved by Cartan when all the data are real-analytic. In the case of complex structures, this partial differential equation expresses the fact that an almost-complex structure is actually a complex structure. In 1957, Newlander and Nirenberg showed that every formally integrable almost-complex structure is integrable, and is in fact a complex structure. Spencer conjectured that the corresponding result would hold for any elliptic Lie pseudogroup; this was proved by Malgrange [19] under the additional assumption that the pseudogroup is analytic. However in 1967, Guillemin and Sternberg [16] gave an example, based on H. Lewy's counterexample to the local solvability of partial differential equations, which shows that the integrability problem is not always solvable. On the other hand, Guillemin and Pollack [21] and Buttin-Molino ([1] and [20]) studied the integrability problem for flat Lie pseudogroups on $\mathbf{R}^n$ and obtained partial results.

In this paper, we give an outline of our proof of the solvability of the integrability problem for all Lie pseudogroups acting on $\mathbf{R}^n$ which contain the translations (Theorem 10), *a fortiori* for all flat pseudogroups. It is a summary of joint work with D. C. Spencer ([12] and [13]) and follows to a large extent Guillemin's program (described in the introduction of [14]) for solving the integrability problem for flat pseudogroups by means of Galois theory type methods similar to those introduced by Sophus Lie in his work on partial differential equations. Guillemin's Jordan-Hölder decomposition for transitive Lie algebras (see [14]) is the required algebraic tool, together with the classification of the simple transitive pseudogroups and the results of Guillemin [14] and Conn [5] on the structure of nonabelian minimal closed ideals of real transitive Lie algebras. We also need the result in several special cases, which is given by the Frobenius, Darboux and Newlander-Nirenberg

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