BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 3, May 1978

EMBEDDINGS OF $(n-1)$-SPHERES IN EUCLIDEAN n-SPACE

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1. Introduction. The program described in this report revolves around the basic embedding question of geometric topology: *under what conditions are two embeddings f_1 and f_2 of a space X in a space Y equivalent in the sense that there exists a self-homeomorphism F of the ambient space Y for which $Ff_1 = f_2$?* This article focuses, in particular, on information and questions concerning embeddings of the $(n-1)$-sphere in Euclidean n-space E^n, a specific problem that serves as a convenient abbreviation for discussing the broader category of embeddings of $(n-1)$-manifolds in n-manifolds, and the article emphasizes a comparison between information about codimension one embeddings in high-dimensional n-manifolds, high usually requiring n to be at least 5, with the extensive collection of known information pertaining to embeddings of 2-manifolds in 3-manifolds.

First, we fix some indispensible notation and conventions. We use B^n to denote the standard n-cell in E^n consisting of all points in E^n having norm ≤ 1 and S^{n-1} to denote the standard $(n-1)$-sphere, also called the boundary ∂B^n of B^n, consisting of all points in E^n having norm $= 1$; we call a space homeomorphic to B^n or S^{n-1} an *n-cell* or an *$(n-1)$-sphere*, respectively. For $1 \leq k < n$ we presume E^k to be included naturally in E^n as the subset whose final $(n-k)$ coordinates each equal 0, thereby determining a standard k-cell B^k and a standard $(k-1)$-sphere S^{k-1} in E^n as well. We use E^k_+ to denote the upper half space in E^k consisting of all points having kth coordinate ≥ 0. We say that a k-cell or a $(k-1)$-sphere X in E^n is *flat* if there exists a homeomorphism of E^n to itself that takes X to the standard object of its type. In this language, our fundamental concern is the question: *under what conditions is an $(n-1)$-sphere in E^n flat?* Equivalently, *under what conditions is an embedding of S^{n-1} in E^n equivalent to the inclusion $S^{n-1} \to E^n$?*

Flatness questions for spheres and cells form the prototype of questions concerning local matters. Let e denote an embedding of an m-manifold M (a metric space locally homeomorphic to either E^m or E^m_+) in the interior of an n-manifold N (henceforth to be written as Int N). One says that e is *locally flat at $x \in M$* (and that $e(M)$ is *locally flat at $e(x)$*) if there exists a

An expanded version of an invited address delivered to the American Mathematical Society at Blacksburg, Virginia, on November 7, 1975; received by the editors February 5, 1977.

AMS (MOS) subject classifications (1970). Primary 57A45, 57A15, 57A35, 57A40; Secondary 57A10, 57C15, 57C30, 57C35.

Key words and phrases. Flat spheres, wild spheres, locally flat embeddings, k-ULC, crumpled n-cubes, sewings, engulfing, decomposition spaces, piercing properties, mapping cylinder neighborhoods.

[1]Research partially supported by NSF Grant GP-33872A3.