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ON THE LATTICE OF FACES OF THE n-CUBE[1]

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1. We give a simple axiom system for the lattice L_n of faces of the n-cube, which is independent of dimension, and we construct a partition of the lattice into a minimum number of chains, or *Dilworth partition.* This partition turns out to enjoy some notable symmetries.

We use the representation of the faces of an n-cube as *signed subsets* of an n-set, say of the set $\{1, 2, \ldots, n\}$. A signed subset $A_\sigma = (A_1, A_2)$ is an ordered pair of disjoint subsets, where A_1 is called the *positive part,* and A_2 the *negative-part.* If B_σ is also a signed set, write $A_\sigma \leqslant B_\sigma$ when $A_1 \supseteq B_1$ and $A_2 \supseteq B_2$. Add a minimum element 0—the *improper face*—to the ordered set of signed sets, thereby making it a lattice L_n. The maximum element I of L_n is the signed set $I = (\phi, \phi)$. We use the terms "face" and "signed set" interchangeably.

On the lattice of signed subsets one defines *diagonals* $\Delta(A_\sigma, \cdot\,)$. For a given face A_σ, such a diagonal is a function defined on the segment $[0, A_\sigma]$ of L_n, and $\Delta(A_\sigma, B_\sigma) = C_\sigma$, where $C_1 = (A_1, B_2)$ and $C_2 = (A_2, B_1)$. On the improper face one sets $\Delta(A_\sigma, 0) = 0$. Geometrically, the diagonal $\Delta(A_\sigma, \cdot\,)$ associates to each face contained in A_σ the unique opposite face inside the face A_σ. When $A_\sigma = I$, the diagonal $\Delta(I, \cdot\,)$, written $\Delta(\,\cdot\,)$, is a cubical analog of complementation in a Boolean algebra.

2. Main Theorem. Let L be a finite lattice with minimum 0 and maximum I. For every $x \neq 0$, let Δ_x be a function defined on the segment $[0, x]$ and taking values in $[0, x]$. Assume: (1) if $y \leqslant x$, then $\Delta_x(\Delta_x(y)) = y$; (2) if $a \leqslant b \leqslant x$, then $\Delta_x(a) \leqslant \Delta_x(b)$; (3) if $a < x$, then $a \wedge \Delta_x(a) = 0$; (4) let $a < x$ and $b < x$. Then the following two conditions are equivalent: $\Delta_x(a) \wedge b < x$ and $a \wedge b = 0$. Then L is isomorphic to the lattice of faces of an n-cube for some n, and conversely.

3. A symmetric Dilworth partition. By Dilworth's theorem there exists a partition of L_n into $[n/3]$ chains. We explicitly describe one such partition, one that is invariant under the main diagonal Δ.

Consider a signed subset as a sequence $u_1 u_2 \cdots u_n$ whose *digits* u_i range over the alphabet $\{x, 0, 1\}$. Set $u_i = x$ if the element i is unsigned; otherwise,

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