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Littlewood-Paley and multiplier theory, by R. E. Edwards and G. I. Gaudry, Springer-Verlag, Berlin, Heidelberg, New York, 1977, ix + 212 pp., $25.60.

Two results, obtained in the early thirties, due to Littlewood and Paley [11], can be considered to be the beginning of the Littlewood-Paley theory. Suppose $f \in L^p(T)$, $1 < p < \infty$, where T is the one-dimensional torus, $c_k = (1/2\pi)\int_{-\pi}^{\pi} f(\varphi)e^{-ik\varphi}\, d\varphi$ and $\sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$ is the Fourier series of f. For $N \geqslant 0$ let

$$\Delta_{\pm(N+1)}(\theta) = \Delta_{\pm(N+1)}(\theta; f) = \sum_{2^N \leqslant \pm k < 2^{N+1}} c_k e^{ik\theta} \tag{1}$$

and $\Delta_0(\theta) \equiv c_0$. The first result is that there exist constants A_p and B_p such that

$$A_p \|f\|_p \leqslant \|d(f)\|_p \leqslant B_p \|f\|_p, \tag{2}$$

where $d(f) = (\sum_{N=-\infty}^{\infty} |\Delta_N|^2)^{1/2}$. When $p = 2$, Plancherel's theorem immediately shows that both of these inequalities are equalities with $A_p = 1 = B_p$. When $p \neq 2$ these inequalities give us a characterization of those trigonometric series that are Fourier series of L^p functions (to wit: $\|(\sum_{-\infty}^{\infty} |\Delta_N|^2)^{1/2}\|_p < \infty$). One of the important features of this characterization is that linear operators obtained by multipliers m_k (of the Fourier coefficients c_k) that vary boundedly on the dyadic blocks Δ_N preserve the class L^p. For example the projection of f onto the trigonometric series of power series type, $\sum_{k=0}^{\infty} c_k e^{ik\theta}$, is immediately seen to be bounded on $L^p(T)$ for $1 < p < \infty$. More generally, the famous Marcinkiewicz theorem stating that for $1 < p < \infty$

$$\begin{aligned} &\left\| \sum_{k=-\infty}^{\infty} m_k a_k e^{ik\theta} \right\|_p \\ &\leqslant A_p \left\{ \left(\sup_{N \geqslant 0} \sum_{2^N \leqslant |k| < 2^{N+1}} |m_{k+1} - m_k| \right) + \sup_k |m_k| \right\} \|f\|_p \end{aligned} \tag{3}$$

is a consequence of (2).

The second result involves the Littlewood-Paley g-function

$$g(f) = \left(\int_0^1 (1 - r)|P_r' * f|^2\, dr \right)^{1/2},$$

where P_r' is the derivative (with respect to r) of the Poisson kernel

$$P_r(\theta) = (1 - r^2)/(1 - 2r\cos\theta + r^2).$$

Again we have inequalities which, like (2), express the equivalence of the L^p-norms of f and $g(f)$ provided $\int_{-\pi}^{\pi} f = 0$: for $1 < p < \infty$ there exist constants A_p and B_p such that

$$A_p \|f\|_p \leqslant \|g(f)\|_p \leqslant B_p \|f\|_p. \tag{4}$$

It turns out that in the original work of Littlewood and Paley the operators mapping f into $d(f)$ were studied by using the properties of the g functions