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Ergodic theory on compact spaces, by Manfred Denker, Christian Grillenberger and Karl Sigmund, Lecture Notes in Math., vol. 527, Springer-Verlag, Berlin, Heidelberg, New York, iv + 360 pp., $13.20.
Ergodic theory and topological dynamics, by James R. Brown, Academic Press, New York, San Francisco, London, x + 190 pp., $19.50.

The term "ergodisch" was coined by Ludwig Boltzmann more than 100 years ago, with a rather intuitive meaning. The first real theorem in ergodic theory was Poincaré's recurrence theorem, proved around 1890. Ironically, the professed anti-Cantorian Poincaré had to wait for the development of rigorous set theoretical measure theory in order to see his theorem fully established. Weyl's equidistribution mod 1 was the second major event in ergodic theory (1916). But in a way all this was prenatal, and the real moment of birth of ergodic theory happened only in 1931, when G. D. Birkhoff and J. v. Neumann proved the individual and mean ergodic theorems, respectively. To be precise, what was born in these years, was measure theoretical ergodic theory, based on the notion of a dynamical system (Ω,B,m,T), where T: $\Omega \to \Omega$ is an m-preservering B-measurable transformation. This is the result of implementing the simple combinatorial framework (Ω,T) with a measure theoretical structure. One could implement it with a topological structure rather than a measure theoretical one choosing a topology (mostly compact metrizable) in Ω and requiring T to be continuous; the result would be what is called topological dynamics. Its origins are largely due to G. D. Birkhoff whose monograph on that subject dates from 1927. Choosing a differentiable structure instead, one arrives at what may be called differentiable dynamics, and is closest to the physical origins of the whole subject. Actually, the physicist would see Ω as the state space of a, say, mechanical system, and T as the result of an evolution over a unit of time. What he wants, e.g., to know is the mean sojourn time

$$\overline{1_E}(\omega) = \lim_{n\to\infty} \frac{1}{n}\left(1_E(\omega) + 1_E(T\omega) + \cdots + 1_E(T^{n-1}\omega)\right)$$

of a state $\omega \in \Omega$ in a set $E \subseteq \Omega$. G. D. Birkhoff's individual ergodic theorem says that

$$\bar{f}(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\omega)$$

exists and is finite m-a.e. if $f \in L_m^1$, for any measure theoretical dynamical system (Ω,B,m,T). The mean ergodic theorem states the same convergence in the sense of L_m^2-mean, for $f \in L_m^2$, and in the L_m^1-mean for any $f \in L_m^1$ in case $m(\Omega) < \infty$. This proves the basic existence statement needed for a rigorous answer to the old physical question whether, in the case of an (Ω,B,m,T) modelling an ideal gas in a box, $\overline{1_E}(\omega)$ is independent of ω and (hence) equals $m(E)/m(\Omega)$. This statement is called the classical ergodic hypothesis. It has been waiting for its full proof for about 100 years. J. G. Sinai sketched a proof in 1963, and the marvelous theory of the Sinai billiard has grown out of attempts to work out the details.

This being the situation with respect to the classical physicists' question, all