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INTERNAL SET THEORY: A NEW APPROACH TO NONSTANDARD ANALYSIS

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1. Internal set theory. We present here a new approach to Abraham Robinson's nonstandard analysis [**10**] with the aim of making these powerful methods readily available to the working mathematician. This approach to nonstandard analysis is based on a theory which we call *internal set theory* (IST). We start with axiomatic set theory, say ZFC (Zermelo-Fraenkel set theory with the axiom of choice [**1**]). In addition to the usual undefined binary predicate $\in$ of set theory we adjoin a new undefined unary predicate *standard.* The axioms of IST are the usual axioms of ZFC plus three others, which we will state below.

All theorems of conventional mathematics remain valid. No change in terminology is required. What is new in internal set theory is only an addition, not a change. We choose to call certain sets standard (and we recall that in ZFC every mathematical object–a real number, a function, etc.–is a set), but the theorems of conventional mathematics apply to all sets, nonstandard as well as standard.

In writing formulas we use $\wedge$ for and, $\vee$ for or, $\sim$ for not, $\Rightarrow$ for implies, and $\Leftrightarrow$ for is equivalent to. We call a formula of IST *internal* in case it does not involve the new predicate "standard" (that is, in case it is a formula of ZFC); otherwise we call it *external.* Thus "x standard" is the simplest example of an external formula. To assert that x is a standard set has no meaning within conventional mathematics–it is a new undefined notion.

The fact that we have adjoined "standard" as an undefined predicate (rather than defining it in terms of $\in$ as is the case with all of the predicates of conventional mathematics) requires a readjustment of an engrained habit. We are used to defining subsets by means of predicates. In fact, it follows from the axioms of ZFC that if $A(z)$ is an internal formula then for all sets x there is a set $y = \{z \in x\colon A(z)\}$ such that for all sets z we have $z \in y \Leftrightarrow z \in x \wedge A(z)$. However, the axioms of ZFC say nothing about external predicates. For example, no axioms allow us to assert that there is a subset S of the set $\mathbf{N}$ of all natural numbers such that for all n we have $n \in S \Longleftrightarrow n \in \mathbf{N} \wedge n$ standard. *We may not use external predicates to define subsets.* We call the violation of this rule illegal set formation.

We adopt the following abbreviations:

$\forall^{\text{st}} x$	for $\forall x(x \text{ standard}) \Rightarrow$,	$\exists^{\text{st}} x$	for $\exists x(x \text{ standard}) \wedge$
$\forall^{\text{fin}} x$	for $\forall x(x \text{ finite}) \Rightarrow$,	$\exists^{\text{fin}} x$	for $\exists x(x \text{ finite}) \wedge$

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