BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

OPTIMIZATION OF THE NORM OF THE LAGRANGE INTERPOLATION OPERATOR

BY T. A. KILGORE

Communicated by Chandler Davis, December 2, 1976

On an interval $[a, b]$, we may place points $t_0, \ldots, t_n$ such that $a = t_0 < t_1 < \cdots < t_n = b$. Using these points, called nodes, we may construct unique polynomials $y_0, \ldots, y_n$ of degree n, such that, for $1 \leqslant i, j \leqslant n$, $y_i(t_i) = 1$ and $y_i(t_j) = 0$ for $j \neq i$. The Lagrange interpolating projection on the nodes $t_0, \ldots, t_n$ is the operator which takes any function f continuous on $[a, b]$ to the polynomial $\Sigma_{i=0}^n f(t_i) y_i$. It is easily seen that this projection is bounded for any degree n, for any interval $[a, b]$, and for any set of nodes in $[a, b]$. The norm is easily shown to be the sup norm of $\Lambda = \Sigma_{i=0}^n |y_i|$, called the Lebesgue function of the projection, and thus the norm depends exclusively on the placement of $t_1, \ldots, t_{n-1}$. It is irrelevant, in attempting to minimize the norm, to move t_0 or t_n. Of the function Λ, it is true that $\Lambda(t_i) = 1$ for $0 \leqslant i \leqslant n$, while if $n \geqslant 2$ and if t is not a node, then $\Lambda(t) > 1$. Let $\lambda_1, \ldots, \lambda_n$ be the values given by

$$\lambda_i = \sup_{t \in [t_{i-1}, t_i]} \Lambda(t) \quad \text{for } 1 \leqslant i \leqslant n.$$

Then $\|\Lambda\| = \max_{1 \leqslant i \leqslant n} \lambda_i$.

It was conjectured by Serge Bernstein in 1932 that the norm of the interpolating projection is minimized when the nodes are so placed that $\lambda_1 = \cdots = \lambda_n$, a conjecture rendered plausible, but by no means demonstrated, by the rather obvious fact that

$$\frac{\partial \lambda_i}{\partial t_i} > 0 > \frac{\partial \lambda_{i+1}}{\partial t_i}, \quad \text{for } 1 \leqslant i \leqslant n-1,$$

and by the fact that moving any node into close proximity with one of its neighbors increases $\|\Lambda\|$ without bound. This communication will give the following theorem and an outline of its proof in a series of lemmas.

THEOREM. *For any $n \geqslant 2$, if the norm of the Lagrange interpolation operator on an interval $[a, b]$ with nodes $a = t_0 < t, < \cdots < t_n = b$ is to be minimized, then it is necessary that the local maximum values $\lambda_1, \ldots, \lambda_n$ of the Lebesgue function be equalized.*

The proof of this theorem depends on the fact that $(\lambda_1, \ldots, \lambda_n)$ is a dif-

---

AMS (MOS) subject classifications (1970). Primary 41A10; Secondary 41A50.