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PARALLEL VECTOR FIELDS AND THE TOPOLOGY OF MANIFOLDS

BY LEON KARP[1]

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0. On a closed, differentiable n-manifold M there exists a nowhere vanishing vector field if and only if the Euler-Poincaré characteristic is zero, $\chi(M) = 0$. If M is riemannian, one may ask whether or not it admits a vector field that is parallel. Chern has shown that necessary conditions are that the first Betti number $b_1 \geqslant 1$ and that the second betti number $b_2 \geqslant b_1 - 1$, and he has conjectured that these conditions are not sufficient [4]. It follows from a special case of a subsequent result of Bott [2] that all the Pontryagin numbers of such a manifold are zero. This is a restriction only if dimension $M \equiv 0 \bmod 4$. The purpose of this note is to announce some further necessary topological conditions. We also exhibit a family of manifolds with $\chi = 0$ that satisfy the conditions of Chern and Bott but still cannot admit parallel vector fields whatever the metric. If the manifold is complex we can refine our results to deduce additional conditions that are necessary for the existence of a Kähler-parallel vector field. These results are applied to the topology of compact homogeneous spaces (supplementing some similar results of Hurewicz and de Rham). Finally, we give some n-dimensional generalizations of some classical results of Hurwitz on Riemann surfaces of genus $\geqslant 2$.

1. Let M^n be an n-dimensional, closed, differentiable manifold and let $b_j(M) \equiv j$th Betti number of M.

THEOREM A. *If M^n admits a vector field that is parallel with respect to some riemannian metric then the Betti numbers of M satisfy*:

$$b_1 \geqslant 1 \quad \textit{and} \quad b_{k+1} \geqslant b_k - b_{k-1}, \quad \textit{for } 1 \leqslant k \leqslant n-1.$$

Notice that when $k = 1$ we have Chern's condition: $b_2 \geqslant b_1 - 1$.

Let M be a closed, complex n-manifold. Then the Hodge number $h^{p,q}(M)$ is the dimension of $H^q(M, \Omega^p)$. For a Kähler manifold it is known that $\Sigma_{p+q=j} h^{p,q}(M) = b_j(M)$ and we can refine Theorem A.

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