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UNICITY OF BEST L_2 APPROXIMATION BY SECOND-ORDER SPLINES WITH VARIABLE KNOTS

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1. Introduction and results. Let $S_N \subset C[0, 1]$ denote the class of all piecewise linear functions with at most $N + 1$ linear segments. In this article we announce some interesting and somewhat surprising approximation properties of S_N in the space $L_2[0, 1]$. Three main theorems will be stated in this section and the main idea of our proof of the first two theorems will be sketched in §2.

Theorem 1 describes a fairly large class of strictly convex functions which have, for each positive integer N, unique best $L_2[0, 1]$ approximants from the nonlinear (spline) manifold S_N. Theorem 2 states that any sufficiently smooth strictly convex function eventually, i.e. for all large N, has a unique best $L_2[0, 1]$ approximant from this manifold. This behavior will be called "eventual uniqueness". Theorem 3 indicates the sharpness of these two results.

We emphasize that S_N is not only a nonlinear manifold, but also a non-closed subset of $L_2[0, 1]$. Hence, arguments regarding existence, uniqueness, and characterization of best approximants are nontrivial. Since it has been shown in [1] that, for every positive integer N, any continuous function has at least one best $L_2[0, 1]$ approximant from S_N, we are only concerned with uniqueness and eventual uniqueness of best approximants in this paper.

THEOREM 1. *Let $f \in C^2[0, 1]$ with $f'' > 0$ on $[0, 1]$. Suppose that $\log f''$ is concave in $(0, 1)$. Then for every positive integer N, f has a unique best $L_2[0, 1]$ approximant from S_N.*

THEOREM 2. *Let $f \in C^5[0, 1]$ with $f'' > 0$ on $[0, 1]$. Then there exists a positive integer N_0 such that for any integer $N > N_0$, f has a unique best $L_2[0, 1]$ approximant from S_N.*

THEOREM 3. *Let N be any positive integer. There exists a function $f \in C^\infty[0, 1]$ with $f'' > 0$ on $[0, 1]$, such that f has more than one best $L_2[0, 1]$ approximant from S_N.*

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