BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

Character theory of finite groups, by I. Martin Isaacs, Pure and Applied Math., Academic Press, New York, San Francisco, and London, 1976, xii + 303 pp., $29.50.

Several approaches to the representation theory of finite groups have been taken in recent works on the subject. Because representation theory has become part of applicable mathematics, there have been books for prospective users in physics, chemistry, and combinatorics. Another point of view was advanced by E. Noether, who observed that the representation theory of finite groups was equivalent to the study of modules over group algebras, and there are books treating group representations as part of the general theory of modules over algebras. A third theme, historically the oldest, is the application of representations and characters to finite group theory, represented in the recent books of Dornhoff [3], Feit [4], Gorenstein [5], Huppert [6], Serre [7], and the book which is the subject of this review, by I. M. Isaacs.

A prospective reader may ask why Isaacs emphasizes character theory rather than representations and modules. The characters, as trace functions of the representations, often yield the most efficient proofs of theorems in group theory. The proofs are usually based on simple, but interesting and often ingenious, computations. In comparison, arguments based on representations and modules, although sometimes conceptually simple, tend to require more machinery.

The first book treating the applications of character theory to group theory is the second edition [1] of Burnside's famous book on group theory, published in 1911. That book contained proofs of the following two results, which have been the inspiration for efforts to widen the scope of the methods used in their proofs up to the present time.

THEOREM A (FROBENIUS). *Let H be a subgroup of a finite group G such that, for all $g \in G - H$, $g^{-1}Hg \cap H = \{1\}$. Then the set of elements N not belonging to any conjugate $g^{-1}Hg$ of H, together with* 1, *form a normal subgroup such that $G = HN$, and $H \cap N = \{1\}$.*

THEOREM B (BURNSIDE). *Let G be a finite group of order p^aq^b, where p and q are primes. Then either G is cyclic of prime order, or G contains a proper normal subgroup. In any case, G is a solvable group, that is, there exist subgroups $G \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_s = \{1\}$, with each G_i normal in G_{i-1}, and each factor group G_{i-1}/G_i abelian.*

Both theorems assert the existence of proper normal subgroups of a finite group. This phenomenon is related to character theory in the following way. Let T: $G \to GL(V)$ be a representation of G by linear transformations on a finite dimensional vector space over the complex field, and let χ: $G \to \mathbf{C}$ be the character of T, given by $\chi(g) = \text{Trace}(T(g))$. Then the kernel N of the homomorphism T, which is a normal subgroup of G, satisfies the condition $N = \{g \in G: \chi(g) = \chi(1)\}$. Conversely, if θ is the character of an arbitrary representation, the set $\{g \in G: \theta(g) = \theta(1)\}$ is always a normal subgroup of G. In this way characters can be used to prove the existence of normal subgroups.

Burnside called attention in [1] to his conjecture that all finite groups of