17. K. Takano and E. Bannai, *A global study of Jordan-Pochhammer differential equations*, Funkcial. Ekvac. **19** (1976), 85–99.

18. M. Yoshida and K. Takano, *On a linear system of Pfaffian equations with regular singular points*, Funkcial. Ekvac. **19** (1976), 175–189.

YASUTAKA SIBUYA

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

Techniques of multivariate calculation, by Roger H. Farrell, Lecture Notes in Math., Springer-Verlag, New York, x + 337 pp., $12.30.

This book is primarily concerned with the mathematical techniques useful in calculating the distribution of functions of random matrices X: $n \times p$ where X has a multivariate normal distribution. As motivation for both this review and much of the material in Farrell's book, I will begin by posing a problem and discussing three possible approaches to solving it. Suppose X is an $n \times p$ random matrix $(n \geq p)$ and X has a density $f(x)$ with respect to Lebesgue measure, l, on the linear space of $n \times p$ matrices. Let $S = X'X \equiv \tau(X) \in \mathcal{S}_p$, where $\mathcal{S}_p$ is the set of all $p \times p$ nonnegative definite matrices (S is positive definite a.e.). The problem is to find the density function of S.

APPROACH 1. Assume that the density $f(X)$ is a function of S as is the case when the elements of X are independent and normal with mean 0 and variance 1. Then $f(X) = g(X'X)$ for some function g. Hence, the density of S is $g(S)$ with respect to the measure $\mu \equiv l \circ \tau^{-1}$ on $\mathcal{S}_p$. All that remains is to calculate the measure μ. Wishart did this in 1928 using a geometric argument which led to the density bearing his name (in the normal case). Of course, $\mu(dS) = c|S|^{(n-p-1)/2}dS$ where c is a constant.

When f is not a function of $X'X$, then the above argument is not available. Two alternative approaches which can be used are now considered.

APPROACH 2. The group $\mathcal{O}(n)$ of $n \times n$ orthogonal matrices acts on the left of X by $X \to \Gamma X$, $\Gamma \in \mathcal{O}(n)$. A maximal invariant function under this action is $\tau(X) = X'X = S$. The density of S with respect to the measure μ given above is q where $q(\tau(x)) = \int f(\Gamma x)\nu(d\Gamma)$. Here, ν is the invariant probability measure on $\mathcal{O}(n)$. This result was used by James (1954) to derive an integral expression for the density function of the noncentral Wishart distribution in the rank 3 case. A result similar to the one above for general compact groups is due to Stein and will be discussed subsequently.

APPROACH 3. Let G_p denote the group of $p \times p$ upper triangular matrices with positive diagonal elements. Also, let $V_{n,p}$ be the set of $n \times p$ matrices ψ which satisfy $\psi'\psi = I_p$. $V_{n,p}$ is called the Stiefel manifold. Each X which has rank p (those with rank less than p have Lebesgue measure 0) can be uniquely written as $X = \psi U$ with $\psi \in V_{n,p}$ and $U \in G_p$. Since $S = X'X = U'U$, a method for finding the density of S is to first find the joint density of ψ and U and then "integrate out" ψ to yield the marginal density of U. With the density of U at hand, the derivation of the density of S is rather routine since the Jacobian of the map $S \leftrightarrow U$ $(S = U'U)$ is easily calculated. To obtain the