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APPLICATIONS OF ANALYSIS ON NILPOTENT GROUPS TO PARTIAL DIFFERENTIAL EQUATIONS[1]

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The last few years have witnessed the birth of a body of techniques for obtaining refined regularity theorems for certain hypoelliptic differential operators through analysis of homogeneous convolution operators on nilpotent Lie groups. (Recall that a differential operator L is *hypoelliptic* if whenever u and f are distributions satisfying $Lu = f$, u must be C^∞ on any open set where f is C^∞.) The purpose of this paper is to present an outline of the development of this theory, beginning with some background and motivation and sketching the main results and the principal highlights of the methods. No proofs will be given, and the theorems will sometimes be stated in less than maximum generality for the sake of brevity.

The principal applications of the theory we are about to discuss occur in the context of two developments of the 1960's: Kohn's work on the $\bar\partial$-Neumann problem and the $\bar\partial_b$ complex, and Hörmander's work on sums of squares of vector fields. We shall therefore give a brief review of these matters, which will serve as motivation and provide us with some terminology to be used later. (The original papers are Kohn **[13]**, **[14]** and Hörmander **[11]**; see also Folland and Kohn **[7]** for a comprehensive exposition of Kohn's work and Kohn **[15]** for a simplified proof of Hörmander's main theorem.)

We first describe Hörmander's theorem. Let M be a C^∞ manifold, and let $X_0, X_1, \ldots, X_n$ be real C^∞ vector fields on M. By a *commutator of order k* of the X_j's we shall mean a vector field of the form

$$[X_{i_1},[X_{i_2},\ldots [X_{i_{k-1}},X_{i_k}]\ldots]], \qquad 0 \leqslant i_j \leqslant n.$$

We shall say that $X_0, \ldots, X_n$ satisfy the *Hörmander condition of order m* if the X_j's and their commutators of order $\leqslant m$ span the tangent space to M at every point.

THEOREM (HÖRMANDER). *If $X_0, \ldots, X_n$ satisfy the Hörmander condition of order m for some $m \geqslant 1$, then the operator*

$$L = \sum_1^n X_j^2 + X_0$$

is hypoelliptic.

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