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ITERATED PATH INTEGRALS[1]

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The classical calculus of variation is a critical point theory of certain differentiable functions (or functionals) on a smooth or piecewise smooth path space, whose differentiable structure is defined implicitly. Because of the importance of path spaces to analysis, geometry and other fields, it is desirable to develop a geometric integration theory or a de Rham theory for path spaces. Having in mind this general goal, we are going to consider a large class of path space differential forms, which can be constructed from usual differential forms by a method of iterated integration.

Recall that the Poincaré lemma is proved through a process of integration, which converts every closed p-form w on a manifold M locally to a $(p-1)$-form. The same process can be used to obtain a $(p-1)$-form $\int w$ defined globally on the total smooth space $P(M)$ of M. More generally, given forms $w_1, \ldots, w_r$ on M, we may repeat the integration process r times in an appropriate manner and obtain a differential form $\int w_1 \cdots w_r$ on $P(M)$ of degree $-r + \Sigma_{1 \leqslant i \leqslant r} \deg w_i$. Such path space differential forms and their linear combinations will be called iterated (path) integrals.

Our objective is to determine the geometrical significance of such iterated integrals. It turns out that they play a surprisingly interesting role in relating analysis on a manifold (or differentiable space) to the homology of its path spaces. For example, Theorem 2.3.1 implies that the real loop space cohomology of a simply connected compact manifold is isomorphic to the cohomology of the complex of iterated integrals as differential forms on the smooth loop space ΩM. Iterated integrals are path space differential forms which permit further integration. They also provide analytic interpretation or realization of algebraic topological notions such as bar constructions [**34**], [**12**], Eilenberg-Moore spectral sequences [**60**], [**61**], Massey products [**46**], [**51**] and loop space cohomology classes of Kraines [**47**].

Our de Rham theoretical approach also produces computational tools, which have the advantage of dealing with commutative differential graded algebras of relatively simple structure. Examples will illustrate how our

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