BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

INVARIANT DIFFERENTIAL EQUATIONS ON HOMOGENEOUS MANIFOLDS

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1. Historical origins of Lie group theory. Nowadays when Lie groups enter in a profound way into so many areas of mathematics, their historical origin is of considerable general interest. The connection between Lie groups and differential equations is not very pronounced in the modern theory of Lie groups, so in this introduction we attempt to describe some of the foundational work of S. Lie, W. Killing and É. Cartan at the time when the interplay with differential equations was significant. In fact, the actual construction of the exceptional simple Lie groups seems to have been accomplished first by means of differential equations.

Although motion groups in $\mathbf{R}^3$ had occurred in the work of C. Jordan prior to 1870, Lie group theory as a general structure theory for the transformation groups themselves originated around 1873 with Lie's efforts about that time to use group theoretic methods on differential equations as suggested by Galois' theory for algebraic equations. It seems that a lecture by Sylow in 1863 (when Lie was 20) on Galois theory[2] (Lie and Engel [**9**, vol. 3, p. XXII]) and his collaboration with F. Klein, 1870, on curves and transformations (Klein and Lie [**6**], Engel [**3b**, p. 35]) were particularly instrumental in suggesting to him the following:

PROBLEM (LIE [**8a**]). *Given a system of differential equations how can knowledge about its invariance group be utilized towards its integration?*

Since the solutions of a differential equation are functions, not just numbers as for an algebraic equation, one can take two different viewpoints for an analogy with Galois theory.

Analytic viewpoint (*Lie* (1871–1874)). For a system of differential equations, consider the group of diffeomorphisms of the underlying space leaving the system stable (i.e., permuting the solutions).

Algebraic viewpoint (*Picard* (1883), *Vessiot* (1891)). For a given differential equation consider the group of automorphisms of the field generated by the solutions, fixing the elements of the coefficient field.

To indicate the flavor of the resulting theories I just recall a couple of the best known results. In ordinary Galois theory one has the fundamental result that an algebraic equation is solvable by radicals if and only if the Galois group is solvable. In the Picard-Vessiot theory one introduces similarly the

An expanded version of an address given at the annual meeting of the American Mathematical Society in Washington, D. C. on January 23, 1975; received by the editors October 20, 1976.

AMS (*MOS*) *subject classifications* (1970). Primary 22E30, 35C15, 43A85, 58G99; Secondary 22-03, 22-02, 43A90.

[1] Partially supported by NSF Grant MCS 75-21415.

[2] See "Sources" at the end of this section.