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CELL-LIKE MAPPINGS AND THEIR GENERALIZATIONS[1]

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Cell-like maps are those whose point-inverses are cell-like spaces (as subspaces of the domain). A space is cell-like if it is homeomorphic to a cellular subset of some manifold. This definition was given in 1968, at which time I began to study proper, cell-like maps between euclidean neighborhood retracts (ENR's). At the time, I pointed out that such maps form a category which includes proper, surjective, contractible maps between polyhedra and proper, cellular maps from a manifold to an ENR. S. Smale had previously studied contractible maps between ANR's, proving a Vietoris-like theorem; M. Cohen had shown that PL contractible maps between finite polyhedra are simple homotopy equivalences; and, of course, proper, cellular maps on manifolds had been studied extensively ("point-like decompositions"). This unification seemed worthwhile at the time because of the equivalence of cell-like (for proper, surjective maps between ENR's) and a condition studied by D. Sullivan in connection with the Hauptvermutung: *the restriction to the inverse of any open set is a proper homotopy equivalence*. Using Sullivan's work it followed that a cell-like map between PL manifolds is often homotopic to a PL homeomorphism.

In 1971, L. C. Siebenmann identified the suspected red herring nature of "PL" in the above sentence: The set of cell-like maps $M \to N$ (where M and N are closed n-manifolds, $n \neq 4$) is precisely the closure of the set of homeomorphisms $M \to N$ in the space of maps $M \to N$. (The case $n = 3$ requires also that M contain no fake cubes and was done earlier by S. Armentrout.)

The "cell-like" concept has since been studied, generalized, and analogized. I will attempt to recount some of this recent work.

Three main topics are identifiable:

A. Finiteness theorems and their global consequences;

B. Mapping cylinder neighborhoods; and

C. Desingularizations of spaces.

Recent major discoveries can be considered at least peripheral to the theme: The topological invariance of simple homotopy type (by T. Chapman), the finiteness of compact ANR's (by J. West), and the locally euclidean nature of the double suspension of certain homology spheres (by R.

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