BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, May 1977

NONUNIQUENESS OF BEST APPROXIMATING COMPLEX RATIONAL FUNCTIONS

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Communicated by Alston S. Householder, November 2, 1976

1. Introduction. For any real continuous function $f(x)$ on $[-1, 1]$ and for any nonnegative integer n, it is well known that the best uniform approximation to $f(x)$ on $[-1, 1]$ by a real rational function of order n is *unique* (cf. [1]). On the other hand, J. L. Walsh [3] has given an example of a continuous complex-valued function on a certain compact crescent-shaped region in the plane, whose best uniform complex rational approximation of order 1 is *not* unique. In this note we present results which show that this nonuniqueness of best complex rational approximations can hold even in the case of approximating *real* functions on finite *real* intervals. The examples given below point out a somewhat unexpected fact; namely, that complex rational functions may yield *closer* uniform approximations to a real function on $[-1, 1]$ than the best real rational function of the same type. This is in marked contrast with the theory for best complex polynomial approximation.

In §2 we will state the main results and examples. The proofs, further details, and related open problems will be published elsewhere [2]. For the remainder of this section we introduce the necessary notation.

For any integer $n \geqslant 0$, we let $\pi^r_{n,n}$ denote the collection of all rational functions which can be written in the form p/q, with p and q polynomials of degree at most n having real coefficients. We use $\pi^c_{n,n}$ to denote the analogous collection of such rational functions with p and q having *complex* coefficients.

If $C_r[-1, 1]$ denotes the collection of all *real* continuous functions on $[-1, 1]$, and if $\|g\| := \sup\{|g(x)|; -1 \leqslant x \leqslant 1\}$ for any real- or complex-valued function g defined on $[-1, 1]$, then we further set

$$E^r_{n,n}(f) := \inf_{g\in\pi^r_{n,n}} \|f-g\|; \qquad E^c_{n,n}(f) := \inf_{g\in\pi^c_{n,n}} \|f-g\|,$$

for any $f \in C_r[-1, 1]$. Obviously, $E^c_{n,n}(f) \leqslant E^r_{n,n}(f)$. The collection of all

AMS (MOS) subject classifications (1970). Primary 41A20; Secondary 41A50.

[1]Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-74-2688.

[2]Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-74-2729, and by the Energy Research and Development Administration (ERDA) under Grant E(11-1)-2075.