BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, May 1977

THE CHOQUET REPRESENTATION IN THE COMPLEX CASE[1]

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1. Introduction and review. In its geometrical form, the Choquet representation theorem can be viewed as an infinite dimensional generalization of a classical theorem of Minkowski concerning finite dimensional compact convex sets. Indeed, suppose that K is a compact convex subset of a locally convex Hausdorff real topological vector space E. If E is assumed to be finite dimensional, then the Minkowski theorem asserts that each point x in K is a convex combination (or *barycenter*) of some finite set of extreme points; that is, there exist positive real numbers $a_1, a_2, \ldots, a_n$ and points $x_1, x_2, \ldots, x_n$ in ext K, the set of extreme points of K, such that $\Sigma a_k = 1$ and $x = \Sigma a_k x_k$. Furthermore, each point of K admits exactly one such representation if and only if K is a simplex. If E is assumed to be infinite dimensional, then the Minkowski theorem fails, although the Krein-Milman theorem does show that such convex combinations of extreme points are dense in K. If K is metrizable, then the Choquet theorem applies (and says more than this): Each point in K is the barycenter (precise definition below) of a Borel probability measure on the G_δ set ext K. Moreover, it is still true that each point of K admits a unique such representation if and only if K is a "simplex" (definition below).

The details of the relationships between the above results, together with some of their applications to real analysis, probability theory, functional analysis, etc., may be found in **[19]**. Additional general references for these and other results left unproved in what follows are **[1]**, **[3]**, **[20]**, **[21]**.

In order to formulate the representation theorem for nonmetrizable compact convex K, we need to introduce some definitions. If X is a compact Hausdorff space, let $M(X)$ denote the space of all complex valued finite regular Borel measures on X. *In what follows, the word "measure" will always mean an element of* $M(X)$. Let $P(X)$ denote the convex set of all probability measures in $M(X)$, i.e., those positive measures of total mass 1. If K is a compact convex subset of the locally convex space E, then for each μ in $P(K)$ there exists a unique point x in K (the *resultant*, or *barycenter* of μ) which is characterized by the relation

$$f(x) = \int_K f\, d\mu \quad (\equiv \mu(f)) \quad \text{for each } f \text{ in } E^*.$$

Equivalently, $h(x) = \mu(h)$ for each h in $A(K)$, the space of all affine real-valued continuous functions on K. If x is the resultant of some μ from $P(K)$, we

An invited address presented at the 710th meeting of the American Mathematical Society, Tucson, Arizona, November 24, 1973; received by the editors December 10, 1975.

AMS (MOS) subject classifications (1970). Primary 28A10, 28A45, 46E15.

Key words and phrases. Compact convex sets, Choquet's theorem, simplexes, affine functions, complex Borel measures.

[1]Work supported in part by a grant from the NSF.