BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 2, March 1977

Embeddings and extensions in analysis, by J. H. Wells and L. R. Williams, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 84, Springer-Verlag, New York, Heidelberg, Berlin, 1975, vii + 108 pp., $14.40.

To embed a given space X into a standard, more distinguished space E, a mapping $\psi: X \to E$ must be exhibited, having certain prescribed properties. In this book E is a Banach space, e.g., a Hilbert space or, more generally, an L^p space, and ψ is an isometry into E. For $E = C[0, 1]$ the existence of such a $\psi: X \to E$, when X is an arbitrary separable metric space, is assured by a rather well-known result of Banach and Mazur. However, because of the strict constraints which are imposed on a metric when it is derived from an inner product, it is a priori clear that no such sweeping result is possible when E is a Hilbert space. Yet by virtue of its elevated position among Banach spaces one would wish to have a better knowledge of the precise conditions a metric space must satisfy to be embeddable in this particular standard space.

The first two chapters of the book provide a detailed answer to this question and to a number of related ones. (Here the contributions of I. Schoenberg, dating back to the late thirties, and those of Schoenberg and J. von Neumann of the early forties are truly impressive and rightly occupy a central position in the presentation.)

Without essentially altering the problem, one considers quasi-metric spaces (X, ρ) (i.e. pairs such that X is a set and ρ is a mapping of $X \times X$ into the nonnegative reals $\mathbf{R}^+$ with $\rho(s, t) = \rho(t, s)$ for all $s,\ t \in X$). A quasi-metric ρ is said to be of negative type if, for every finite subset $\{x_0, \ldots, x_n\}$ of X,

$$\sum_{j,k=0}^{n} \rho(x_j, x_k)^2 \xi_j \xi_k \leqslant 0 \quad \text{whenever} \sum_{j=0}^{n} \xi_j = 0.$$

It turns out that for a quasi-metric space to be (isometrically) embeddable into a sufficiently large Hilbert space it is necessary and sufficient that ρ be of negative type.

If F: $\mathbf{R}^+ \to \mathbf{R}^+$ is continuous, $F(0) = 0$, and $F \circ \rho$ is of negative type, then F itself is said to be of negative type and the collection of all such functions is denoted by $N(X)$. A related class of functions, called radial positive definite, consists of all continuous F: $\mathbf{R}^+ \to \mathbf{R}^+$ with the property that

$$\sum_{j,k=1}^{n} F(\rho(x_j, x_k))\xi_j \xi_k \geqslant 0$$

for all finite subsets $\{\xi_1, \ldots, \xi_n\}$ of the reals and all finite sets $\{x_1, \ldots, x_n\}$ in X; it is denoted by $RPD(X)$.

There is an intimate connection between $N(X)$ and $RPD(X)$ as

$$F \in N(X) \Leftrightarrow \exp(-\lambda F^2) \in RPD(X) \qquad (\lambda > 0).$$

Clearly then any information on $RPD(X)$ is interpretable in terms of embeddings, via $N(X)$, and vice versa. For example, by expressing $|t|^{2\alpha}$ in terms of the integral $\int_0^\infty (1 - \exp(-\lambda^2 t^2))\lambda^{-1-2\alpha}\, d\lambda$, it is possible to show that