BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

A STRONG NONCOMMUTATIVE ERGODIC THEOREM

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Communicated by R. G. Bartle, June 5, 1976

Let α be an automorphism of a von Neumann algebra $\mathfrak{A}$ such that $\mathfrak{A}$ has a faithful α-invariant normal state ρ. For A in $\mathfrak{A}$ and $n = 1, 2, \ldots,$ write

$$s_n(A) = \frac{1}{n}\, [A + \alpha(A) + \alpha^2(A) + \cdots + \alpha^{n-1}(A)].$$

The result referred to in the title is the following.

THEOREM 1. *For each A in $\mathfrak{A}$ and $\epsilon > 0$ there exist an α-invariant element $\hat{A}$ and a projection E in $\mathfrak{A}$ such that $\rho(E) > 1 - \epsilon$ and*

$$\|(s_n(A) - \hat{A})E\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

We describe the convergence by saying that $s_n(A) \longrightarrow \hat{A}$ "almost uniformly."

Consider the case where $\mathfrak{A}$ is commutative. Then there is a probability measure space (X, S, μ) such that $\mathfrak{A}$ is isomorphic to $L^\infty(X, \mu)$ in such a way that ρ corresponds to the functional $f \longmapsto \int_X f\, d\mu$. Also, α corresponds to an automorphism of $L^\infty(X, \mu)$ which is induced by an invertible measure-preserving transformation of X. It is easy to see (using Egorov's theorem) that almost uniform convergence in $\mathfrak{A}$ corresponds to almost everywhere convergence on X. So in this setting the above theorem reduces to the classical pointwise ergodic theorem of G. D. Birkhoff [1] (for a bounded function f).

The first ingredient in the proof of Theorem 1 is the theorem of Kovács and Szücs [2], which establishes the existence of the conditional expectation mapping $A \longmapsto \hat{A}$, and can be adapted to show that $s_n(A) \longrightarrow \hat{A}$ strongly. This result can be regarded as a noncommutative version of the mean ergodic theorem. Next, it is necessary to show the existence of a large set of elements A in $\mathfrak{A}$ for which we can take the projection E to be the identity.

THEOREM 2. *Let $\mathfrak{A}_u = \{A \in \mathfrak{A}\colon \|s_n(A) - \hat{A}\| \longrightarrow 0\}$. For any A in $\mathfrak{A}$ there is a bounded sequence (A_k) in $\mathfrak{A}_u$ which converges strongly to A.*

The proof of Theorem 2 involves some Fourier analysis and the spectral theorem for a unitary operator. If we write $B_k = A - A_k$ then $B_k \longrightarrow 0$ strongly and so $s_n(B_k) \longrightarrow 0$ strongly as $k \longrightarrow \infty$, for each n. If we knew that this convergence was, in a suitable sense, uniform in n, we could complete the proof of Theorem 1 by von Neumann algebra techniques. The desired uniformity is provided by the following theorem.

AMS (MOS) subject classifications (1970). Primary 46L10; Secondary 28A65, 46A05, 47A35.

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