BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

EXISTENCE THEOREMS ACROSS A POINT OF RESONANCE

BY LAMBERTO CESARI

Communicated by Robert G. Bartle, April 8, 1976

1. Let us consider the equation

$$Ex + \alpha x = Nx, \tag{1}$$

where E: $D(E) \longrightarrow S$, $D(E) \subset S$, is a linear not necessarily bounded operator in a real Hilbert space S with $1 \leqslant \dim W < \infty$, $W = \ker E$, and N: $S \longrightarrow S$ is a continuous nonlinear operator. In terms of the alternative method (see Cesari [1], [2]), let P: $S \longrightarrow S$ be the orthogonal projector with range $PS = S_0 = W = \ker E$, let $S_1 = (I - P)S$, and let S_1 be also the range of E. Let H: $S_1 \longrightarrow S_1$ denote a linear, bounded, compact operator (a partial inverse of E) such that the usual relations of the alternative method (selfadjoint case) hold: (h_1) $H(I - P)E = I - P$; (h_2) $EP = PE$; (h_3) $EH(I - P) = I - P$. Let $(\ ,\ )$ and $\|\ \|$ denote inner product and norm in S, and let $L = \|H(I - P)\|$.

I (An existence theorem "at resonance"). *Under the above hypotheses, let us assume that*

(B) *there is a constant* $J_0 > 0$ *such that* $\|Nx\| \leqslant J_0$ *for all* $x \in S$; *and*

(N_0) *there is a constant* $R_0 \geqslant 0$ *such that* $(Nx, x^*) \leqslant 0$ [*or always* $(Nx, x^*) \geqslant 0$] *for all* $x \in S$, $x^* \in S_0$ *with* $Px = x^*$, $\|x^*\| \geqslant R_0$, $\|x - Px\| \leqslant LJ_0$.

Then the equation $Ex = Nx$ *has at least a solution* $x \in X$.

We refer to Cesari and Kannan [6] for a proof of I by Schauder's fixed point theorem, and to Kannan and McKenna [8] for a proof based on a Leray-Schauder argument. It has been pointed out that the hypotheses contained in recent specific and relevant theorems by Landesman and Lazer [9], Williams [13], Nečas [11], and Lazer and Leach [10] concerning existence at resonance for boundary value problems for ordinary and partial differential equations, imply conditions (B) and (N_0). But the same hypotheses of the aformentioned specific theorems also imply the stronger condition (N_ϵ) below. In turn, this condition has a stronger implication which we present here.

II (An existence theorem "across a point of resonance"). *Under the same general hypotheses above, let us assume that* (B) *holds, and that*

AMS (MOS) subject classifications (1970). Primary 47H15, 34B15, 34C15, 35G30, 35J40.