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DUALITY AND KATO'S THEOREM ON SMALL PERTURBATIONS

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ABSTRACT. The main result is a theorem on stability of index under small perturbations in locally convex spaces, which reduces for Banach spaces to the familiar theorem of T. Kato.

It is a well-known result of Gohberg and Kreĭn [2] and Kato [3] that if T is a semi-Fredholm operator and P a bounded operator of norm small enough, then $T + P$ is a semi-Fredholm operator with the same index as T. Kato gives a precise upper bound of the norm of P: $\|P\| < \gamma(T)$, where $\gamma(T)$ essentially is $\|\hat{T}^{-1}\|^{-1}$, $\hat{T}$ being the one-to-one operator induced by T. In geometric terms, this may be expressed as $TB \supset \lambda B' \cap R(T)$, $PB \subset \mu B'$ and $0 \leqslant \mu < \lambda$, B, B' being the unit balls of E, F.

Some results concerning small bounded perturbations of Φ_--operators in more general locally convex spaces are given in [7], [4], but they do not fully render the precise Kato theorem in case of Banach spaces.

By using Kato's theorem in the dual, we obtain some results (Propositions 1 and 3) which do constitute an extension of Kato's theorem on small perturbations of Φ_--operators, and refine several results in [7], [4].

In the sequel, E, F always denote two Hausdorff locally convex spaces, and T, P two (linear) operators from E into F such that $[D(T)]^- \subset D(P)$, $[D(T)]^-$ being the closure of the domain of T. Let $N(T)$ and $R(T)$ denote the kernel and the range of T. By neighborhood we mean an absolutely convex neighborhood of the origin. A *disk* is an absolutely convex set.

The operator T is *open* (resp. *almost open*) if TU (resp. $[TU]^-$) is a neighborhood in $R(T)$, for any neighborhood $U \subset E$. T is a Φ_- (resp. Φ_+)-*operator* if T is open, has a closed graph (in $E \times F$) and a closed range, and $\operatorname{codim} R(T) < \infty$ (resp. $\dim N(T) < \infty$). The *index* of T is then defined as $\operatorname{ind}(T) = \dim N(T) - \operatorname{codim} R(T)$ (we do not distinguish between different cardinalities of infinity).

PROPOSITION 1. *Let T be an almost open operator with* $\operatorname{codim}[R(T)]^- < \infty$, *and P a continuous operator.*

(1) *Assume that there exists a base of neighborhoods U in E such that*

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