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THE INVARIANTS OF $n \times n$ MATRICES

BY CLAUDIO PROCESI

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The purpose of this paper is threefold: to classify the invariants of matrices, study their relations, and understand the quotient varieties that these invariants classify (i.e. to determine when two sets of matrices have the same invariants). We obtain fairly complete answers to these questions. The groups with respect to which we study invariants are the classical algebraic groups $\mathrm{GL}(n, \mathbf{C})$, $O(n, \mathbf{C})$, $\mathrm{Sp}(2n, \mathbf{C})$ and their maximal compact subgroups. The action on m-tuples of matrices is always the diagonal action with conjugation in each component. We list the main results:

(1) The invariants of matrices $X_1, X_2, \ldots, X_i, \ldots$ under $\mathrm{GL}(n, \mathbf{C})$ are generated by the monomials $\mathrm{Tr}(X_{i_1} X_{i_2} \cdots X_{i_s})$. More generally the matrix valued invariants are generated by these monomials and the variables X_i.

(2) Every relation among such invariants is a consequence of the Hamilton-Cayley Theorem.

For the case of $\mathrm{GL}(n, \mathbf{C})$ the study of the quotient variety was carried out by A. Artin, who proved that two m-tuples of matrices $(X_1, \ldots, X_m)$, $(Y_1, \ldots, Y_m)$ have the same invariants if and only if the "semisimple parts" (i.e. associated semisimple representations) are conjugate.

For the unitary group $U(n, \mathbf{C})$ one has:

(3) The invariants are generated by the monomials $\mathrm{Tr}(U_{i_1} U_{i_2} \cdots U_{i_s})$ where $U_i = X_i$ or $U_i = \overline{X}_i^t$.

(4) Every relation is a consequence of the Hamilton-Cayley Theorem.

(5) Two sets of matrices $(X_1, \ldots, X_i, \ldots)$, $(Y_1, \ldots, Y_i, \ldots)$ are conjugate under $U(n, \mathbf{C})$ if and only if they have the same invariants.

(6) The invariants can be given arbitrarily subject only to:
(a) The formal algebraic relations and
(b) The inequalities $\mathrm{Tr}(p\overline{p}^t) \geqslant 0$, p a polynomial in $X_i, \overline{X}_i^t$.

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