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INDECOMPOSABLE MODULES: AMALGAMATIONS

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The determination of criteria for amalgamations (pushouts) of indecomposable modules to be indecomposable is a well-known central problem of representation theory upon which not much progress has been made. Here we announce two results on amalgamations, and describe some of their applications to the representation theory of Artin rings. We refer the reader to [3] for the notions of modules with cores and modules with cocores, recalling that such modules are indecomposable. In the cited paper basic modules were also discussed—we mention that a module B of finite length is basic precisely when $B/\mathrm{rad}\, B$ is simple. Our first result is valid in any module category.

THEOREM 1. *Let M_1 be indecomposable and $\theta_i\colon A \longrightarrow M_i$ proper monomorphisms, $i = 1, 2$, such that* $\mathrm{Hom}(M_1, M_2/\mathrm{im}\, \theta_2) = 0$. *Let X be the pushout of θ_1 and θ_2.*

(i) *If M_2 is indecomposable and* $\mathrm{Hom}(M_2, M_1/\mathrm{im}\, \theta_1) = 0$, *then X is indecomposable.*

(ii) *If $M_2/\mathrm{im}\, \theta_2$ is indecomposable then X is indecomposable if and only if there is no homomorphism $f\colon M_2 \longrightarrow M_1$ such that $f\theta_2 = \theta_1$.*

The only result we know of bearing any resemblance to our second result is a lemma of Ringel [5, p. 313].

THEOREM 2. *If*

$$0 \longrightarrow S \longrightarrow B_1 \oplus B_2 \xrightarrow{(\rho_1, \rho_2)} M \longrightarrow 0$$

is a nonsplit exact sequence where S is a simple module and B_1 and B_2 are basic modules of finite length, then M is indecomposable if and only if neither ρ_1 nor ρ_2 is a split monomorphism.

If S is a simple submodule of a nonsimple basic module B over a homomorphic image of a hereditary left Artin ring such that $\mathrm{Ext}^1(B, B) = 0$ then, using Theorem 2, it is readily checked that the cokernel of some homomorphism $S \longrightarrow B \oplus B$ is indecomposable if and only if $\mathrm{Ext}^1(B/S, B) \neq 0$. When $\mathrm{Ext}^1(B, B) \neq 0$ we have

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