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INDEPENDENT KNOTS IN BIRKHOFF INTERPOLATION[1]

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We consider Birkhoff interpolation for an incidence matrix $E = (e_{ik})_{i=1}^{m}{}_{;}{}_{k=0}^{n}$, the "polynomials" $P = \Sigma_0^n a_k u_k(x)$, for a system $U = \{u_k\}_0^n$ of functions $u_k \in C^n[a, b]$ (or $P = \{x^k\}_0^n$) and the knots $X = (x_1, \ldots, x_m)$ satisfying $a \leqslant x_1 < \cdots < x_m \leqslant b$. The method of independent knots appears for the first time in [4]; it is somewhat related to the coalescence method [1], [3].

A function $f \in C^n[a, b]$ is annihilated by E, X if

$$f^{(k)}(x_i) = 0 \quad \text{for all } (i, k) \text{ with } e_{ik} = 1. \tag{1}$$

From zeros of f and its derivatives given by (1), one can derive further zeros by means of Rolle's theorem. This leads to the following definition. A *Rolle set* $\mathcal{R}$ for a function f annihilated by E, X is a collection $\mathcal{R}_k$, $k = 0, \ldots, n$, of Rolle sets of zeros (with multiplicities) of the $f^{(k)}$. The sets $\mathcal{R}_k$ are defined inductively: $\mathcal{R}_0$ consists of the zeros of f given by (1); if $\mathcal{R}_0, \ldots, \mathcal{R}_k$ have been defined, we select $\mathcal{R}_{k+1}$—some of the zeros of $f^{(k+1)}$—as follows: (α) $\mathcal{R}_{k+1}$ contains all zeros of $f^{(k)}$ of multiplicity > 1, their multiplicities reduced by 1. (β) $\mathcal{R}_{k+1}$ contains all zeros of $f^{(k+1)}$ (with multiplicities) given by (1). (γ) For any two adjacent zeros $\alpha, \beta \in \mathcal{R}_k$ we select a zero γ of $f^{(k+1)}$ by means of Rolle's theorem, *provided one exists not listed in* (1). This new zero γ may be different from the x_i; it may be one of the x_i, but not listed in (1) as a zero of $f^{(k+1)}$; finally, γ may appear as an additional multiplicity of a zero x_i of $f^{(k+1)}$ by (1). In this case, $e_{i,k+1} = \cdots = e_{i,k+t} = 1$, $e_{i,k+t+1} = 0$. If no zero γ as specified exists, there is a *loss*. (δ) We adjust the multiplicities in the last case of (γ): if also $e_{i,k+t+2} = \cdots = e_{i,k+s+1} = 0$, then γ belongs to $\mathcal{R}_{k+1}$ with multiplicity s. A Rolle set constructed without losses is *maximal*. A function f annihilated by E, X may have several Rolle sets, some of them maximal, others are not. Let m_k be the number of ones in the column k of E, let

$$\mu_k = (\cdots((m_0 - 1)_+ + m_1 - 1)_+ + \cdots + m_{k-1} - 1)_+ + m_k. \tag{2}$$

LEMMA 1. *The number of distinct Rolle zeros of $f^{(k)}$ in a maximal Rolle set is exactly μ_k.*

Let E be a Birkhoff matrix, let E^0 be derived from E by replacing a one,

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