GENERATORS FOR ALGEBRAS OF RELATIONS

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Let \( B_n \) denote the collection of all binary relations on the set \( X = \{ 1, 2, \ldots, n \} \). The purpose of this paper is to observe that there exists a pair of relations on \( X \) that generate all of \( B_n \) under the boolean operations and relational composition.

In [1] C. J. Everett and S. M. Ulam introduced the notion of an abstract projective algebra. McKinsey [2] showed that every projective algebra is isomorphic to a subalgebra of a complete atomic projective algebra and thus, in view of the representation given in [1], every projective algebra is isomorphic to a projective algebra of subsets of a direct product; that is, to an algebra of relations.

**Projective Algebra.** A boolean algebra \( P \) with unit \( 1 \) and zero \( 0 \), so that for all \( x \in P \), \( 0 \leq x \leq 1 \), is said to be a projective algebra if there are defined two mappings \( \pi_1 \) and \( \pi_2 \) of \( P \) into \( P \) satisfying the following:

\[
P_1. \quad \pi_i(a \lor b) = \pi_i a \lor \pi_i b.
\]

\[
P_2. \quad \pi_1 \pi_2 1 = p_0 = \pi_2 \pi_1 1 \text{ where } p_0 \text{ is an atom of } P.
\]

\[
P_3. \quad \pi_1 a = 0 \text{ if and only if } a = 0.
\]

\[
P_4. \quad \pi_i \pi_i a = \pi_i a.
\]

\[
P_5. \quad 0 < a \leq \pi_1 1, 0 < b \leq \pi_2 1, \text{ there exists an element } a \otimes b \text{ such that } \pi_1 (a \otimes b) = a, \pi_2 (a \otimes b) = b, \text{ with the property that } x \in P, \pi_1 x = a, \pi_2 x = b \text{ implies } x \leq a \otimes b.
\]

\[
P_6. \quad \pi_1 1 \otimes p_0 = \pi_1 1; p_0 \otimes \pi_2 1 = \pi_2 1.
\]

\[
P_7. \quad 0 < u, v \leq \pi_1 1 \text{ implies } (x \lor y) \otimes \pi_2 1 = (x \otimes \pi_2 1) \lor (y \otimes \pi_2 1); \text{ and } 0 < u, v \leq \pi_2 1 \text{ implies } \pi_1 1 \otimes (u \lor v) = (\pi_1 1 \otimes u) \lor (\pi_1 1 \otimes v).
\]

If the projective algebra \( P \) is a complete atomic boolean algebra, then \( P \) is called a complete atomic projective algebra. The projective algebra \( P \) is said to be projectively generated by a subset \( A \) if \( P \) can be obtained from \( A \) using \( \pi_1, \pi_2, \otimes \) and the boolean operations.

Consider \( B_n \) and let \( p_0 = (1, 1) \). We define the mappings \( \pi_1, \pi_2 : B_n \rightarrow B_n \) and a product \( \otimes : B_n \times B_n \rightarrow B_n \) as follows:

(i) \( \pi_1 \alpha = \alpha((X \times X)p_0), \)

(ii) \( \pi_2 \alpha = (p_0(X \times X))\alpha, \)

(iii) \( \alpha \otimes \beta = (\alpha(X \times X))\beta, \)

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