BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

GENERATORS FOR ALGEBRAS OF RELATIONS[1]

BY A. R. BEDNAREK AND S. M. ULAM

Communicated by J. T. Schwartz, May 11, 1976

Let $\mathcal{B}_n$ denote the collection of all binary relations on the set $X = \{1, 2, \ldots, n\}$. The purpose of this paper is to observe that there exists a *pair* of relations on X that *generate* all of $\mathcal{B}_n$ under the boolean operations and relational composition.

In [1] C. J. Everett and S. M. Ulam introduced the notion of an abstract projective algebra. McKinsey [2] showed that every projective algebra is isomorphic to a subalgebra of a complete atomic projective algebra and thus, in view of the representation given in [1], every projective algebra is isomorphic to a projective algebra of subsets of a direct product; that is, to an algebra of relations.

PROJECTIVE ALGEBRA. A boolean algebra $\mathcal{P}$ with unit 1 and zero 0, so that for all $x \in \mathcal{P}$, $0 \leqslant x \leqslant 1$, is said to be a projective algebra if there are defined two mappings π_1 and π_2 of $\mathcal{P}$ into $\mathcal{P}$ satisfying the following:

P_1. $\pi_i(a \vee b) = \pi_i a \vee \pi_i b$.

P_2. $\pi_1\pi_2 1 = p_0 = \pi_2\pi_1 1$ where p_0 is an atom of $\mathcal{P}$.

P_3. $\pi_i a = 0$ if and only if $a = 0$.

P_4. $\pi_i\pi_i a = \pi_i a$.

P_5. For $0 < a \leqslant \pi_1 1$, $0 < b \leqslant \pi_2 1$, there exists an element $a \square b$ such that $\pi_1(a \square b) = a$, $\pi_2(a \square b) = b$, with the property that $x \in \mathcal{P}$, $\pi_1 x = a$, $\pi_2 x = b$ implies $x \leqslant a \square b$.

P_6. $\pi_1 1 \square p_0 = \pi_1 1$; $p_0 \square \pi_2 1 = \pi_2 1$.

P_7. $0 < x, y \leqslant \pi_1 1$ implies $(x \vee y) \square \pi_2 1 = (x \square \pi_2 1) \vee (y \square \pi_2 1)$; and $0 < u, v \leqslant \pi_2 1$ implies $\pi_1 1 \square (u \vee v) = (\pi_1 1 \square u) \vee (\pi_1 1 \square v)$.

If the projective algebra $\mathcal{P}$ is a complete atomic boolean algebra, then $\mathcal{P}$ is called a *complete atomic projective algebra.* The projective algebra $\mathcal{P}$ is said to be *projectively generated* by a subset $\mathcal{A}$ if $\mathcal{P}$ can be obtained from $\mathcal{A}$ using π_1, π_2, $\square$ and the boolean operations.

Consider $\mathcal{B}_n$ and let $p_0 = (1, 1)$. We define the mappings π_1, π_2: $\mathcal{B}_n \longrightarrow \mathcal{B}_n$ and a product $\square$: $\mathcal{B}_n \times \mathcal{B}_n \longrightarrow \mathcal{B}_n$ as follows:

(i) $\pi_1\alpha = \alpha((X \times X)p_0)$,

(ii) $\pi_2\alpha = (p_0(X \times X))\alpha$,

(iii) $\alpha \square \beta = (\alpha(X \times X))\beta$,

AMS (MOS) subject classifications (1970). Primary 02J10.

[1]Research supported by NSF Grant No. MSC75-21130.