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A FIXED POINT THEOREM FOR PLANE HOMEOMORPHISMS

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The purpose of this note is to outline a proof of "every homeomorphism of the plane into itself that leaves a continuum M invariant has a fixed point in $T(M)$". That is, the orientation preserving condition in the Cartwright Littlewood fixed point theorem [3] is unnecessary.

All sets will be assumed to be subsets of the plane unless otherwise indicated.

DEFINITION. If A is a bounded set then $T(A)$ is the smallest compact set that contains A and has a connected complement.

THEOREM 1. *Let $f: D \longrightarrow R^2$ be a map defined on a simple closed curve D. If there is a partition of D, $\{x_0, x_1, x_2, \ldots, x_n = x_0\}$ and arcs $A_1, A_2, A_3, \ldots, A_n$ in $T(D)$ such that A_i joins $f(x_{i-1})$ to $f(x_i)$ and $x_{i-1}x_i \cap T(f[x_{i-1}x_i] \cup A_i) = \emptyset$, then every extension of f to a map defined on $T(D)$ has a fixed point.*

PROOF. Suppose there is a fixed point free extension of f to a map g defined on $T(D)$. Then find mutually disjoint (except for endpoints) arcs $K_1, K_2, \ldots, K_n$ in $T(D)$ such that K_i joins x_{i-1} to x_i and $T(K_i \cup x_{i-1}x_i) \cap T(f(K_i) \cup A_i) = \emptyset$. Then using the Tietze extension theorem, piece together a map g': $T(D) \longrightarrow R^2$ for which $g'(z) = g(z)$ if $z \notin \bigcup \{T(x_{i-1}x_i \cup K_i): i = 1, 2, \ldots, n\}$, $g'(x_{i-1}x_i) \subset A_i$, and $g'(T(x_{i-1}x_i \cup K_i)) \cap T(x_{i-1}x_i \cup K_i) = \emptyset$. If r is a retract of R^2 onto

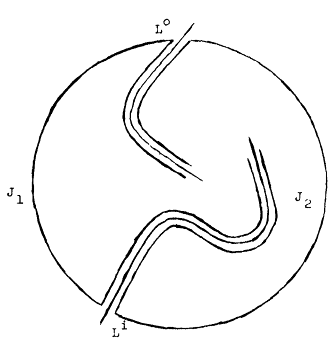

FIGURE 1

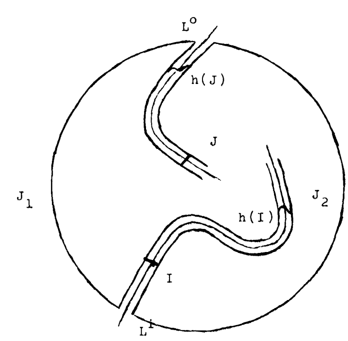

FIGURE 2

AMS (MOS) subject classifications (1970). Primary 54H25, 55C20.

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