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THEORY OF ANNIHILATION GAMES

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Throughout, $R = (V(R), E(R))$ is a finite loopless digraph with vertex set $V(R)$ and edge set $E(R) \subset V(R) \times V(R)$, which may contain cycles. Let $F(u) = \{v \in V(R): (u, v) \in E(R)\}$, Z = nonnegative integers, $GF(2)^n$ = the n-fold cartesian product of $GF(2)$.

Put any number of stones on distinct vertices of R. Two players play alternately. Each player at his turn moves one stone from a vertex u to some $v \in F(u)$. If v was occupied, both stones get removed (*annihilation*). The player making the last move wins. If there is no last move, the game is a tie.

Such an *annihilation game* belongs to a large class of combinatorial games discussed in [1], [3], which are analyzable by the *Generalized Sprague-Grundy Function* (GSG-function) G: $V(R) \longrightarrow Z \cup \{\infty\}$ [1], [2], [3] with associated counter function c: $V^f(R) \longrightarrow Z$, where $V^f(R) = \{u \in V(R): G(u) < \infty\}$ [2]. Here R is the *game-graph* of the game.

Our main result is the construction of a complete strategy for the game, which is polynomial in $n = |V(R)|$.

Let $C(R)$ be the game-graph of the annihilation game on R, also called the *contrajunctive compound* of R. If $V(R) = \{u_1, \ldots, u_n\}$, the vertices of $V(C(R))$ (= game positions) constitute the set of all n-tuples $\bar{u} = (\alpha_1, \ldots, \alpha_n)$ over $GF(2)$, where $\alpha_i = 1$ if and only if a stone is on u_i. Also $(\bar{u}, \bar{v}) \subset E(C(R))$ if and only if there is a move from $\bar{u}$ to $\bar{v}$. Thus $V(C(R))$ is identical with the linear space $GF(2)^n$ under the operation $\oplus$, Σ' of *Nim-sum* (below: *Generalized Nim-sum* [1], [3]).

LEMMA 1. *Let*

$$C^f(R) = \{\bar{u} \in V(C(R)): G(\bar{u}) < \infty\}, \quad C_i(R) = \{\bar{u} \in V(C(R)): G(\bar{u}) = i < \infty\}.$$

Then

(i) *$C^f(R)$ is a linear subspace of $V(C(R))$.*

(ii) *G is a homomorphism from $C^f(R)$ onto $GF(2)^t$ with kernel $C_0(R)$ ($t = O(\log_2 n)$). In fact,*

$$G(\bar{u}) < \infty \Rightarrow G(\bar{u} \oplus \bar{v}) = G(\bar{u}) \oplus G(\bar{v}).$$

(iii) $\{C_i(R): 0 \leqslant i < 2^t\} = C^f(R)/C_0(R)$.

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