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THE REGULARITY OF ELLIPTIC AND PARABOLIC FREE BOUNDARIES

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In this report we shall sketch a proof of the fact certain free boundaries in R^n are smooth. This result can be applied to the variational solutions of obstacle problems, filtration problems and the one phase Stefan problem for melting ice, which have recently been obtained by various authors [1], [3], [4].

We first consider the following localization of the weak solution of an elliptic free boundary problem: We are given an open set W, a linear elliptic operator $Au = \Sigma a_{ij}(x)\partial_i\partial_j$ ($a_{ij} \in C^3$ in a neighborhood of $\overline{W}$), and a function $v \in C^{1,1}(W)$ and satisfying: (1) $v \geqslant 0$, $Av = f$, where f has a C^α ($\alpha > 0$) extension f^* to a neighborhood of $\overline{W}$, with $f^* \geqslant \lambda > 0$; (2) $\partial W = \partial_1(W) \cup \partial_2(W)$ where $\partial_1 W$ is open in ∂W and $v = |\nabla v| = 0$ on $\partial_1 W$. ($\partial_1 W$ is a part of the free boundary.) F will denote an open subset of $\partial_1 W$ with $\overline{F} \subset \partial_1 W$.

THEOREM 1. *If $X_0 \in F$ is a nonzero density point for the complement CW of W, there is a ball $B_\rho(X_0)$ of radius ρ, centered at X_0, such that $F \cap B_\rho(X_0)$ is the graph of a C^1 function and $v \in C^2((W \cup F) \cap B_\rho(X_0))$.*

REMARK. This result has two virtues: first it shows that the variational solution is a classical one; second, it then follows from unpublished results of D. Kinderlehrer and L. Nirenberg that the gradient of the free boundary (as the graph of a function) is as differentiable as f and a_{ij}.

The proof goes as follows: First, Lemma 1, we prove that the pure second derivatives, v_{ii}, of v do not remain negative near F. More precisely, for $X \in W$, $v_{ii}(X) > -C|\log d(X, F)|^{-\epsilon}$, where $d(X, F)$ is the distance from X to F and $\epsilon > 0$. The geometric consequence of this fact is that, if $Y \in W$, $v(Y) > \rho^2$, and $d(Y, F) < \rho^{1/2}$, then there exists a half ball,

$$HB(Y, C\rho\,|\log \rho\,|^{\epsilon'}) = B(Y, C\rho\,|\log \rho\,|^{\epsilon'}) \cap \{X\colon \langle X - Y, \eta\rangle \geqslant 0\}$$

which is contained in W. If we recall that for $X \in \overline{W}$, $\sup_{B_\rho(X)\cap W} v \geqslant C\rho^2$, this lemma provides for each $X \in \overline{W}$ a half ball contained in W, whose radius is much larger than the distance between its center and X.

The following step, Lemma 2, controls how rapidly CW must become

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