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# STABILITY AND GROWTH ESTIMATES FOR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS IN HILBERT SPACE

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Let $H$, $H_+$ be real Hilbert spaces with $H_+$ dense in $H$ and $H_+ \subset H$, algebraically and topologically; the inner products on $H$, $H_+$ are denoted by $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_+$, respectively. As in [1], let $H_-$ denote the dual of $H_+$ via the inner product of $H$ so that $H_-$ is the completion of $H$ under the norm

$$\|\mathbf{w}\|_- = \sup_{v \in H_+} \frac{|\langle v, w\rangle|}{\|v\|_+}.$$

By $L(H_+, H_-)$ we denote the space of bounded linear operators from $H_+$ to $H_-$. For $0 \leqslant t < T$, $T > 0$ an arbitrary real number, we consider the initial-value problem

(1) $$\mathbf{u}_{tt} - \mathbf{N}\mathbf{u} + \int_{-\infty}^{t} \mathbf{K}(t-\tau)\mathbf{u}(\tau)\,d\tau = \mathbf{0},$$

(2) $$\mathbf{u}(0) = \mathbf{f}, \qquad \mathbf{u}_t(0) = \mathbf{g},$$

where $\mathbf{N} \in L(H_+, H_-)$ is symmetric and $\mathbf{K}(t), \mathbf{K}_t(t) \in L^2((-\infty, \infty); L(H_+, H_-))$. We also assume that

(3) $$\mathbf{u}(\tau) = \mathbf{U}(\tau), \qquad -\infty < \tau < 0,$$

where $\mathbf{U}(t) \in C^1((-\infty, 0); H_+)$ is prescribed and satisfies $\lim_{t\to 0^-} \mathbf{U}(t) = \mathbf{f}$, $\lim_{t\to 0^-} \mathbf{U}_t(t) = \mathbf{g}$, $\lim_{t\to -\infty} \|\mathbf{U}(t)\|_+ = 0$ and $\int_{-\infty}^{0} \|U(t)\|_+\,dt < \infty$.

In [2] we have proved the following basic result concerning solutions $\mathbf{u} \in C^2([0, T); H_+)$ for which $\mathbf{u}_t \in C^1([0, T); H_+)$ and $\mathbf{u}_{tt} \in C([0, T); H_-)$. Let

$$\mathcal{N} = \left\{ \mathbf{w} \in C^2([0, T); H_+) \,\Big|\, \sup_{[0,T)} \|\mathbf{w}(t)\|_+ \leqslant N^2 \right\}$$

for some real number $N$. Then we have

THEOREM (BLOOM [2]). *Let $\mathbf{u} \in \mathcal{N}$ be any solution of (1)–(3) and define*

$$F(t; \beta, t_0) = \|\mathbf{u}(t)\|^2 + \beta(t + t_0)^2, \qquad 0 \leqslant t < T,$$

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