BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, July 1976

DISCONNECTED SOLUTIONS

BY W. F. LUCAS[1]

Communicated by James H. Bramble, February 20, 1976

1. **Introduction.** In the book, *Theory of games and economic behavior* (1944), J. von Neumann and O. Morgenstern introduced a theory of solutions (or stable sets) for multi-person cooperative games in characteristic function form. A longstanding conjecture has been that the *union* of all solutions of any particular game is a connected set. (E.g., see [3].) This announcement describes a twelve-person game for which the conjecture fails. The essential definitions for an n-person game will be reviewed briefly before the counterexample is presented. A sketch of the proof is presented here, and the details will appear elsewhere.

2. **The model.** An *n-person game* is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ is the set of *players* and v is a *characteristic function* on 2^N, i.e., v assigns the real number $v(S)$ to each subset S of N and $v(\varnothing) = 0$. The set of *imputations* is

$$A = \left\{x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geqslant v(\{i\}) \text{ for all } i \in N\right\}$$

where $x = (x_1, x_2, \ldots, x_n)$ is a vector with real components. For any $S \subset N$, let $x(S) = \Sigma_{i \in S} x_i$. For any $X \subset A$ and nonempty $S \subset N$, define $\text{Dom}_S X$ to be the set of all $x \in A$ such that there exists a $y \in X$ with $y_i > x_i$ for all $i \in S$ and with $y(S) \leqslant v(S)$. Let $\text{Dom}\, X = \bigcup_{\phi \neq S \subset N} \text{Dom}_S X$. A subset V of A is a *solution* if $V \cap \text{Dom}\, V = \varnothing$ and $V \cup \text{Dom}\, V = A$. The *core* of a game is

$$C = \{x \in A: x(S) \geqslant v(S) \text{ for all nonempty } S \subset N\}.$$

For any solution V, $C \subset V$ and $V \cap \text{Dom}\, C = \varnothing$.

A characteristic function v is *superadditive* if $v(S \cup T) \geqslant v(S) + v(T)$ whenever $S \cap T = \varnothing$. The game below does not have a superadditive v as is assumed in the classical theory, but it is equivalent solutionwise to a game with a superadditive v. (See [1, p. 68].)

3. **Example.** The 13 vital coalitions for our example consist of $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and elements from three classes:

$\mathcal{B} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$,

---

AMS (MOS) subject classifications (1970). Primary 90D12.

Key words and phrases. Game theory, solutions, stable sets, cores, characteristic functions, domination relations.

[1]Research supported in part by NSF grant MPS75-02024 and ONR contract N00014-75-C-0678.