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# THE DIMENSION OF ALMOST SPHERICAL SECTIONS OF CONVEX BODIES

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The well-known theorem of Dvoretzky [1] states that convex bodies of high dimension have low dimensional sections which are almost spherical. More precisely, the theorem states that for every integer $k$ and every $\epsilon > 0$ there is an integer $n(k, \epsilon)$ such that any Banach space $X$ with dimension $\geqslant n(k, \epsilon)$ has a subspace $Y$ of dimension $k$ with $d(Y, l_k^2) \leqslant 1 + \epsilon$. Here $d(Y, l_k^2)$ denotes the Banach Mazur distance coefficient between $Y$ and the $k$ dimensional Hilbert space $l_k^2$ i.e. $\inf\|T\|\ \|T^{-1}\|$ taken over all operators $T$ from $Y$ onto $l_k^2$.

The estimate for $n(k, \epsilon)$ given in [1] was improved in [5] to $n(k, \epsilon) = e^{c(\epsilon)k}$.

In other words (considering the dependence of $n(k, \epsilon)$ on $k$ for fixed $\epsilon$) the dimension of the almost spherical section (of the unit ball) given by Dvoretzky's theorem is about the log of the dimension of the space. This estimate is in general the best possible, since as observed in [10] it is easy to verify that if $X = l_n^\infty$ any subspace $Y$ of $X$ whose Banach Mazur distance from a Hilbert space is $\leqslant 2$ say, must be of dimension at most $C \log n$. It turns out however that if we take into account not only $X$ but also $X^*$ much better information can be obtained.

THEOREM 1. *There is an absolute constant $c > 0$ so that for every Banach space $X$ of dimension $n$ the following formula holds*

$$k(X)k(X^*) \geqslant cn^2\|P\|^2/d^2(X, l_n^2). \tag{1}$$

This formula needs explanation. The number $k(X)$ is an integer with the following property: "most" subspaces $Y$ of $X$ of dimension $k(X)$ satisfy $d(Y, l_{k(X)}^2) \leqslant 2$. We do not intend to make the word "most" precise here. What really matters is that there exists a $Y \subset X$ with $d(Y, l_{k(X)}^2) \leqslant 2$. The number $\|P\|$ is the norm of a projection $P$. If $k(X) \leqslant k(X^*)$, $P$ is a projection from $X$ onto a suitable subspace $Y$ of $X$ with $d(Y, l_{k(X)}^2) \leqslant 2$ (again actually "most" subspaces $Y$ of dimension $k(X)$ work). If $k(X^*) \leqslant k(X)$, $P$ is a projection from $X^*$ onto a subspace $Z \subset X^*$ with $d(Z, l_{k(X^*)}^2) \leqslant 2$.

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