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DEFORMATIONS OF GEODESIC FIELDS

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We describe here a deformation theorem for geodesic fields on a Riemannian manifold and an obstruction whose vanishing is necessary and sufficient for deforming one geodesic field into another. As an application we prove that every smooth manifold of dimension $\geqslant 2$ can be given a Riemannian metric with a nontriangulable cut locus. In [5] we obtain an "equivariant" deformation theorem by entirely different methods, and a consequent strengthening of the cut locus results. We thank Professors Richard Hamilton and Albert Nijenhuis for insights gained in conversation with them.

1. The cut locus. On each geodesic from the point p on the compact Riemannian manifold M, the *cut point* is the last point to which the geodesic minimizes distance, and the *cut locus* $C(p)$ is the set of these. This notion was introduced by Poincaré in 1905 and has since played an important role in global differential geometry [3].

In 1935 Myers [4] showed that on a compact analytic surface, the cut locus can be triangulated as a finite graph. Recently this has been extended to arbitrary dimensions by the work of Buchner, Mather, Hironaka and Kato [2]. Buchner proved in his thesis [1] that on any smooth compact manifold of dimension $\leqslant 5$, there is an open and dense set of Riemannian metrics for which the cut locus of a point p is structurally stable under perturbation of metric (and one expects triangulable). By contrast we prove

THEOREM 1. *On any smooth manifold of dimension* $\geqslant 2$, *there is a Riemannian metric and a point* p *with nontriangulable cut locus* $C(p)$.

2. The connection with deformations of geodesic fields. The problem of preassigning a cut locus led us to the study of deformations of geodesic fields. In Figure 1 we start with a round sphere and try to preassign a cut locus C of the south pole. We draw out from C a smooth family G' of geodesics heading roughly south, and from the south pole the family G of geodesics heading north. It seems natural to try to deform the round metric in a neighborhood of the equator so as to deflect the geodesic field G' into G, like a lens focusing some light rays from C to make them converge at the south pole.

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